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# Methodology of measuring performance in alternative investment

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## Abstract

The development of alternative investment has highlighted the limitations of standard performance measures like the Sharpe ratio, primarily because alternative strategies yield returns distributions which can be far from gaussian. In this paper, we propose a new framework in which trades, portfolios or strategies of various types can be analysed regardless of assumptions on payoff. The proposed class of measures is derived from natural and simple properties of the asset allocation. We establish representation results which allow us to describe our set of measures and involve the log-Laplace transform of the asset distribution. These measures include as particular cases the squared Sharpe ratio, Stutzer's rank ordering index and Hodges' Generalised Sharpe Ratio. Any measure is shown to be proportional to the squared Sharpe ratio for gaussian distributions. For non gaussian distributions, asymmetry and fat tails are taken into account. More precisely, the risk preferences are separated into gaussian and non-gaussian risk aversions.

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## 1 Introduction

Measuring the performance or choosing the optimal allocation in an alternative investment context raises several difficulties. First, fund managers are facing a large choice of strategy and payoff types, resulting in the observation of a vast array of potential return distributions, for example exhibiting asymmetry, fat tails, or even multimodality characteristics. Despite its natural appeal as a simple, intuitive tool, the Sharpe ratio is not suitable when the shape of the return distribution is far from normal. The need for a measure adapted to any kind of distribution is then particularly clear for alternative investment. Another limitation of standard methods in this context is that they are based on returns, which is not always adapted to the positions taken in alternative investment. For example for a single trade as simple as a future, the notion of return is not really well defined and cannot correctly take account of the leverage. Note that in addition, an ad hoc method is needed to cope with short sales, a common tool in hedge funds strategies.

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To remedy this, we propose a new framework in which trades, portfolios or strategies of various types can be analysed regardless of assumptions on payoff. In particular, it allows a fund manager to compare and work out his optimal allocation. In this framework, we describe a position as its final net worth (including transaction costs and properly discounted cash flows), modelised as a random variable  $X$  on a probability space  $(\Omega, \mathcal{F}, P)$ . We work with prices rather than returns, since this choice will be more appropriate in the alternative investment context. This allows us to evaluate the performance of a portfolio having an initial value of 0. However, when limited to a unique period of time (as in this paper), our methodology can be directly transposed to returns.

In further studying the desirable properties of a performance indicator in the context of alternative investment, we develop an axiomatic description of an adapted performance measure. It is based on two axioms concerning the optimal portfolio achievable with two independent trades. We observe first that the squared Sharpe ratio, Stutzer's rank ordering index, and Hodges' Generalised Sharpe Ratio satisfy these axioms. Then we prove that a performance measure satisfying these axioms always coincides with the squared Sharpe ratio on gaussian variables.

Introducing a natural regularity assumption, we give then a full characterization of our class of measures. We prove that any measure satisfying the assumptions can be expressed for any random variable  $X$  as:

$$(1.1) \quad \pi(X) = \sup_{\lambda \in \mathbb{R}} \mathcal{J}(H_{\lambda X})$$

where  $\mathcal{J}$  is linear and continuous and  $H_X$  denotes the log-Laplace transform: for  $\lambda \in \mathbb{R}$ ,  $H_X(\lambda) = -\ln E(e^{-\lambda X})$ . In this paper, we limit our study to distributions  $X$  having a proper Laplace transform, i.e. such that  $\{\lambda \in \mathbb{R} | E(e^{-\lambda X}) < +\infty\}$  contains 0 in its interior. This corresponds to rather thin tails for unbounded random variables. This work can be extended to fatter tails as those found in stable laws. In (1.1), the application  $J : X \mapsto \mathcal{J}(H_X)$  appears itself as a performance measure and  $\pi(X)$  can be interpreted as the best value achieved by varying the quantity of  $X$ . Therefore the measure  $\pi$  refers to a potentiality, which will only be realized if the asset is hold in the right quantity. This generalizes Hodges' approach, where the exponential utility function is maximized.

The structure of the paper is as follows. After a brief description of some known measures, we describe in section 2 what properties a performance measure should satisfy. Section 3 presents the first consequences of the axioms on the measures, in particular their value for gaussian variables. The representation result (1.1) is proved in section 4. Under an additional financial assumption for digital contracts, the associated measure  $J$  is expressed, in section 5, as the action of a Schwartz distribution of order at most two on the log-Laplace transform<sup>3</sup>. Section 6 proposes the interpretation in terms of risk aversion. In the whole paper, the squared Sharpe ratio and the Hodges measure (detailed in section 1.3), serve as simple illustrative examples. Section 7 compares the portfolio optimization obtained with these two examples and a more complex measure, while section 8 summarizes the most important results, and also gives an insight on the results of a subsequent paper which will be devoted to further properties of these measures. We conclude with some topics for future research.

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<sup>3</sup>Note that a reciprocal property of this result is proved in Appendix C.

## 1.1 Sharpe ratio and classical alternatives

Standard performance measurement and portfolio optimization have long been based on the Sharpe ratio. The use of the latter relies on the assumption that investors choose portfolios according to a mean-variance framework. It is well known that the Sharpe ratio is particularly misleading when the return distribution is far from normal (see e.g. Bernardo and Ledoit [12], Goetzmann and al. [17] or Till [29]). Using variance as the risk measure is indeed not appropriate when investment strategies have asymmetric or leptokurtic outcomes. This is the case with option strategies, and for example, option sellers score particularly high Sharpe ratios. In [17], Goetzmann and al. derive an optimal option-based strategy for achieving the maximum expected Sharpe ratio. This strategy has a truncated right tail and a fat left tail. That means that “expected returns being held constant, high Sharpe ratio strategies” are those that “generate modest profits punctuated by occasional crashes”. Because option-like characteristics appear sometimes in hedge funds performances<sup>4</sup>, this confirms that the Sharpe ratio will be of limited use in this context. We will test our new family of measures on the same example of portfolio choice.

Some extensions have been proposed using risk measures such as downside deviation (Sortino ratio), VaR, CVaR or higher moments. Pedersen and Satchell [22] propose a survey of risk measures and remark that several have been converted into equilibrium risk measures in asset pricing. For example, Bawa and Lindenberg [11] derive alternative CAPM-models based on lower partial moments.

## 1.2 Stutzer’s rank ordering index

Stutzer [28] bases his performance measure on the minimization of the probability that the growth rate of invested wealth will not exceed an investor-selected target growth rate. He chooses a portfolio that makes this probability decay to zero asymptotically, as the time horizon  $T \rightarrow \infty$ , with a decay rate which is as high as possible. A simple result in large deviations theory is used to show that this decay rate maximization criterion is equivalent to maximizing an expected power utility of the ratio of invested wealth to the benchmark, with a risk aversion parameter determined by maximization. If we denote by  $R$  the log return in excess of the benchmark, this criterion leads, in the i.i.d. case, to choose portfolios which maximize  $\sup_{\lambda} [-\ln E(e^{-\lambda R})]$  (see [28]).

When the difference in the log gross returns of portfolios and benchmark is gaussian, this criterion is half the squared Sharpe ratio and yields then the same ranking as the Sharpe ratio for portfolios with a positive Sharpe ratio.

## 1.3 Hodges measure

Hodges [18] introduces a performance measure based on the exponential utility function  $U(W) = -e^{-\alpha W}$ . It is presented as a generalization of the Sharpe ratio, since it reduces to it for gaussian outcomes. The Generalised Sharpe Ratio (GSR) is defined as a measure of market opportunities. Its value on a given asset  $X$  satisfies:  $\frac{1}{2}GSR^2(X) = -\ln[-U^*(X)]$

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<sup>4</sup>In [29], Till give several examples of alternative strategies where the investors are implicitly short of options.

where  $U^*(X)$  is the optimal expected utility obtainable when the investor chooses the best level of investment (or sale) in the considered asset:  $U^*(X) = \sup_{\lambda \in \mathbb{R}} [-E(e^{-\alpha\lambda X})]$ . The measure of an opportunity is then obtained by maximising on the quantity  $\lambda$  invested in it. Since it uses a utility function with constant absolute risk aversion, the composition of the optimal portfolio is independent of the coefficient of risk tolerance  $\alpha$ . Moreover the measure provides rankings which are consistent with stochastic dominance.

We observe that Hodges and Stutzer approaches yield similar expressions. However Hodges bases his analyses on prices, while Stutzer uses log returns (on prices, it corresponds to a constant relative risk aversion utility function, with an endogenous risk aversion parameter). Since we are working with prices, we will be referring to Hodges' measure when using this criterion in the remaining of the paper and we set:

$$(1.2) \quad Hod(X) = \sup_{\lambda \in \mathbb{R}} [-\ln E(e^{-\lambda X})]$$

Hodges' approach has been followed for example by Madan and McPhail [21] who use a slightly modified measure to develop a ranking statistic based on the four first moments. The higher moments are calibrated to distributions using the variance gamma class of processes.

## 1.4 Omega measure

Keating and Shadwick [20] propose a performance measure which is the ratio of average gain on average loss, where gains as well as losses are considered with respect to a given benchmark:  $\Omega(\rho) = \frac{E[(X-\rho)^+]}{E[(\rho-X)^+]}$  measures the quality of a bet on a return  $> \rho$ . It is a measure of the relative probability weighted gains to losses at the return level  $\rho$ . For two assets, the one with the higher Omega is a better bet. The derivation of this measure does not rely on any assumption about risk aversion. Note that giving  $\rho \mapsto \Omega(\rho)$  is equivalent to setting the distribution of  $X$ . For a given  $\rho \in \mathbb{R}$ ,  $\Omega(\rho) - 1$  has a usual form of risk/return criterion since it is the ratio of the mean excess return over  $\rho$  on the first lower partial moment  $E[(\rho - X)^+]$ , used as risk measure.

Bacmann and Pache [10] compare portfolio optimization according to Stutzer' index and the Omega measure with traditional mean-variance framework, on empirical series of hedge funds indices returns. They find that the optimal weights of indices associated with negative skewness and high kurtosis are reduced when compared to the mean-variance approach. Moreover, the portfolios optimised with the new measures provide better out-of-sample returns than the ones constructed in the mean-variance approach.

## 2 Description of our framework

### 2.1 Axioms related to the allocation among independent trades

A position (or asset, or trade) is a given risk exposure, considered as a gain opportunity. All cash-flows linked to a position are taken into account at the end of a given trading period and properly discounted, including premia, transaction costs and payoff of the asset itself. The

relevant variable is then the discounted net profit-and-loss (P&L) at the end of the trading period. It can be either the realized or the forecast P&L (depending if we use the measure for comparison of funds, considering their past performances, or as an asset allocation criterion), for a trade, a fund, a part of a fund... This final net worth is described as a random variable on a probability space  $(\Omega, \mathcal{F}, P)$ . We denote by  $\mathcal{X}$  the set of variables  $X$  on the probability space  $(\Omega, \mathcal{F}, P)$  such that  $\{\lambda \in \mathbb{R} | E(e^{-\lambda X}) < +\infty\}$  contains 0 in its interior. A performance measure will be an application on  $\mathcal{X}$ , with values in  $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ . We make the classical assumption of *Law Invariance*, meaning that the performance of  $X$  depends only on its law<sup>5</sup>.

We consider performance measures  $\pi : \mathcal{X} \rightarrow \overline{\mathbb{R}}$  having the following properties: the optimal portfolio obtained with two independent trades has a measure equal to the sum of the measures of each separate trade (A1), and in this optimal portfolio, the weight of a trade depends only on its own characteristics and not on the other trades (A2).

### Assumptions (A)

(A1) For all  $X, Y \in \mathcal{X}$  independent,

$$(2.1) \quad \sup_{\alpha, \beta \in \mathbb{R}} \pi(\alpha X + \beta Y) = \pi(X) + \pi(Y)$$

(A2) There exists an “optimal weight” function  $X \in \mathcal{X} \mapsto \lambda_X \in \overline{\mathbb{R}}$  such that:

$\lambda_X$  depends only on the law of  $X$ ,  $\lambda_1 = \lambda_{\mathcal{N}(1,1)}$  is positive<sup>6</sup> and

$\forall X, Y \in \mathcal{X}$  independent, the upper bound in (2.1) is achieved for  $\alpha = \lambda_X$  and  $\beta = \lambda_Y$ , possibly asymptotically for an infinite weight.

(AT) We exclude  $\pi \equiv 0$  on the set of gaussian variables and we assume:

- (i)  $\forall X \in \mathcal{X}$ ,  $\exists Z$  gaussian independent of  $X$ , such that  $\pi(Z + X) \in \mathbb{R}$ .
- (ii)  $x \mapsto \pi(\mathcal{N}(x, x))$  is bounded on a non empty open interval.

Note that the weights of the trades have values in  $\mathbb{R}$ , meaning that assets can be bought or short sold. We do not impose that the weights sum to 1 in a portfolio, since no limit is set on the positions and all cash-flows are taken into account at the end of the period only. (A2) gives the optimal allocation between two independent trades. ‘Asymptotically achieved’ means, for example if  $\lambda_X = +\infty$  and  $\lambda_Y \in \mathbb{R}$ , that we have:

$$\sup_{\alpha, \beta \in \mathbb{R}} \pi(\alpha X + \beta Y) = \lim_{\alpha \rightarrow +\infty} \pi(\alpha X + \lambda_Y Y)$$

We do not assume that the optimal weight is unique, and  $X \mapsto \lambda_X$  will denote below any weight function such that assumption (A2) holds<sup>7</sup>. The condition  $\lambda_1 > 0$  is a natural requirement: it means that  $\mathcal{N}(1, 1)$  will be strictly preferred to  $\mathcal{N}(-1, 1)$ , ensuring that  $\pi$  corresponds to a performance measure.

<sup>5</sup>It allows us to use  $\pi(\mathcal{L})$  instead of  $\pi(X)$  if  $\mathcal{L}$  is the law of the variable  $X$ .

<sup>6</sup>We use  $\lambda_{\mathcal{L}}$  for  $\lambda_X$  if  $X$  has law  $\mathcal{L}$ .

<sup>7</sup>In particular, we could take any real for  $\lambda_0$ . We will always choose  $\lambda_0 = 0$ .

(AT) is a technical assumption preventing pathological candidates for  $\pi$ . An example is given in section 2.2.1 where condition (i) is not satisfied.

## 2.2 Examples

These assumptions are satisfied by the following measures on  $\mathcal{X}$ .

### 2.2.1 Squared Sharpe ratio

As the other measures, the Sharpe ratio will be calculated on the final net worth of the trades. We denote by  $Sh(X)$  the Sharpe ratio of  $X$ . Let us consider two independent random variables  $X$  and  $Y$  in  $\mathcal{X}$  and compute

$$(2.2) \quad \sup_{\alpha, \beta \in \mathbb{R}} Sh^2(\alpha X + \beta Y)$$

If  $V(X)$  and  $V(Y)$  are positive, (2.2) is achieved for  $(\alpha, \beta)$  proportional to  $(\frac{E(X)}{V(X)}, \frac{E(Y)}{V(Y)})$  and is worth  $Sh^2(X) + Sh^2(Y)$ . To ensure  $\lambda_{\mathcal{N}(1,1)} = \lambda_1$ , we set  $\lambda_X = \lambda_1 \frac{E(X)}{V(X)}$  for  $V(X) \neq 0$ . If  $V(X) = 0$ , with  $X$  positive, for any  $Y \in \mathcal{X}$  with  $V(Y) \neq 0$ , we have  $\lim_{\alpha \rightarrow +\infty} Sh^2(\alpha X + \lambda_Y Y) = +\infty$ , then (2.2) is achieved asymptotically for  $\alpha \rightarrow +\infty$ ,  $\beta = \lambda_Y$  and is worth  $+\infty$ , as is  $Sh^2(X) + Sh^2(Y)$ . Therefore  $Sh^2$  satisfies (A1) and (A2), with the optimal weight function  $X \mapsto \lambda_X = \lambda_1 \frac{E(X)}{V(X)}$  (including infinite values if  $V(X) = 0$ ).

For  $Z \sim \mathcal{N}(1, 1)$  and  $X \in \mathcal{X}$ ,  $Sh^2(Z + X) = \frac{(1+E(X))^2}{1+V(X)}$  is finite, while  $Sh^2(\mathcal{N}(x, x)) = x$  for  $x > 0$ , which prove (AT).

Note that without assumption (AT) (i), the following (undesirable) measure

$$\pi(X) = \begin{cases} +\infty & \text{if } X \text{ has moments with degree at least 3 which are non null} \\ Sh^2(X) & \text{else} \end{cases}$$

would satisfy assumptions (A).

### 2.2.2 Hodges measure

According to (1.2), Hodges' measure can be written by mean of the log-Laplace transform of the variables. This function will also be involved in the characterization of our family of measures.

**Definition 2.1** For any random variable  $X$ , we set  $D_X = \{\lambda \in \mathbb{R} | E(e^{-\lambda X}) < +\infty\}$  and define the function  $H_X$  on  $\mathbb{R}$  by:

$$H_X(\lambda) = \begin{cases} -\ln E(e^{-\lambda X}) & \text{if } \lambda \in D_X \\ -\infty & \text{else} \end{cases}$$

When  $X \in \mathcal{X}$ , the distribution of  $X$  is uniquely determined by  $H_X$  (see section 4.1 for properties of the log-Laplace). With this definition, we have:

$$\mathcal{X} = \{X \mid 0 \in \overset{\circ}{D_X}\}$$



where  $\overset{\circ}{D}$  denotes the interior of the set  $D$ , and (1.2) can be written  $Hod(X) = \sup_{\lambda \in \mathbb{R}} H_X(\lambda)$ .

We have

$$(2.3) \quad H_{X+Y} = H_X + H_Y \quad \text{for } X, Y \in \mathcal{X} \text{ independent}$$

since then  $\forall \lambda \in \mathbb{R}$ ,  $E(e^{-\lambda(X+Y)}) = E(e^{-\lambda X})E(e^{-\lambda Y})$ . Therefore

$$\sup_{\alpha, \beta \in \mathbb{R}} \sup_{\lambda \in \mathbb{R}} [-\ln E(e^{-\lambda(\alpha X + \beta Y)})] = \sup_{\alpha \in \mathbb{R}} [-\ln E(e^{-\alpha X})] + \sup_{\beta \in \mathbb{R}} [-\ln E(e^{-\beta Y})]$$

Thus  $\sup_{\alpha, \beta} Hod(\alpha X + \beta Y) = Hod(X) + Hod(Y)$  and the upper bound is achieved<sup>8</sup> for  $(\alpha, \beta) = (\lambda_X, \lambda_Y)$  where  $\lambda_Z$  denotes an element of  $\overline{\mathbb{R}}$  where  $H_X$  is maximum (concave function). Then (A1) and (A2) hold with the weight function  $X \mapsto \lambda_X$ .

We give more details on the finiteness of  $Hod(X)$  in section 4.2 and appendix A. In particular, proposition 8.3 proves that  $Hod(Z + X)$  is finite for any  $X \in \mathcal{X}$ , when  $Z$  is gaussian and non constant. Then all assumptions (A) are satisfied.

## 2.3 Regularity

To achieve the characterization of our family of performance measures, we will need a regularity assumption on the measures.

**Definition 2.2** Let  $\mathcal{H}$  denote the set of log-Laplaces of variables of  $\mathcal{X}$ :  $\mathcal{H} = \{H_X \mid 0 \in \overset{\circ}{D}_X\}$ . To any  $\pi : \mathcal{X} \rightarrow \overline{\mathbb{R}}$ , we associate a function on  $\mathcal{H}$  by setting, for  $H_X \in \mathcal{H}$ ,  $\Pi(H_X) = \pi(X)$ .

Note that when  $X \in \mathcal{X}$ ,  $H_X$  is  $C^\infty$  on  $\overset{\circ}{D}_X$ .

**Definition 2.3** For  $I$  interval of  $\mathbb{R}$ , let  $C^\infty(I)$  denote the space of  $C^\infty$  functions on  $I$  and let  $C^\infty = \bigcup \{C^\infty(I) \mid I \text{ open interval}\}$ . We define on  $C^\infty$  a family of semi-norms indexed by  $(K, p)$ , for  $K$  compact  $\subset \mathbb{R}$  and  $p \in \mathbb{N}$  by setting, for  $H \in C^\infty$ :

$$\|H\|_{K,p} = \begin{cases} \max_{0 \leq k \leq p} \max_{\lambda \in K} |H^{(k)}(\lambda)| & \text{if } H \text{ is } C^\infty \text{ on } K \\ +\infty & \text{else} \end{cases}$$

**Notation 2.4** For  $x \geq 0$ , let  $Z_x$  denote a random variable with law  $\mathcal{N}(2x\lambda_1, 2x\lambda_1^2)$ .

Let us consider  $\pi : \mathcal{X} \rightarrow \overline{\mathbb{R}}$ , satisfying assumptions (A). Roughly speaking, we would like to express, as a regularity requirement, the derivability of  $\Pi$  at  $H_{Z_1}$ . However, excepted in special cases like  $X$  divisible and  $\frac{1}{\varepsilon} \in \mathbb{N}$ , in general  $H_{Z_1} + \varepsilon H_X$  is not the log-Laplace of another distribution, and  $\Pi(H_{Z_1} + \varepsilon H_X)$  is not defined. Then, instead of considering  $\frac{\Pi(H_{Z_1} + \varepsilon H_X) - \Pi(H_{Z_1})}{\varepsilon}$ , we will study  $\Pi(H_{Z_x} + H_X) - \Pi(H_{Z_x})$  with  $x = \frac{1}{\varepsilon}$ , for a reason which will appear in section 4.3 (see (4.3)).

Note that in this paper, we will consider only independent variables, in particular expressions like  $\pi(X + Y)$  will always refer to independent  $X$  and  $Y$ . This will be sufficient to define and characterize the family of measures.

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<sup>8</sup>Includes the case asymptotically achieved for an infinite weight.

**Notation 2.5** For  $\pi : \mathcal{X} \rightarrow \overline{\mathbb{R}}$ , we denote by  $\mathcal{X}_\pi$  the set of variables  $X \in \mathcal{X}$  such that, for  $\lambda$  in a neighborhood of 1,  $\lim_{n \rightarrow +\infty} [\pi(Z_n + \lambda X) - \pi(Z_n)]$  exists<sup>9</sup> and is in  $\mathbb{R}$ . We set  $\mathcal{H}_\pi = \{H_X | X \in \mathcal{X}_\pi\}$ .

**Notation 2.6** For  $K \subset \mathbb{R}$  and  $\eta > 0$ , we set :  $K^\eta = \bigcup_{\lambda \in [1-\eta, 1+\eta]} \lambda K$ .

### Assumption (R)

*Lipschitz condition near gaussian variables, with respect to the family of semi-norms*

$$\left| \begin{array}{l} \exists p \in \mathbb{N}, K \text{ compact} \subset \bigcap_{X \in \mathcal{X}_\pi} D_X^\circ \text{ and } M > 0 \text{ such that } \forall X_1, \dots, X_k \in \mathcal{X}_\pi, \\ \exists \eta_0 > 0, \forall \eta \leq \eta_0, \exists M_\eta > 0 \text{ such that } \forall H_Y, H_{Y'} \in \text{Vect}_{\mathbb{N}}(H_{X_1}, \dots, H_{X_k}), \\ n \geq M_\eta \max(1, \|H_Y\|_{K,p}, \|H_{Y'}\|_{K,p}) \Rightarrow |\pi(Z_n + Y) - \pi(Z_n + Y')| \leq M \|H_Y - H_{Y'}\|_{K^\eta, p}. \end{array} \right.$$

For this condition to be active, we assume that  $\mathcal{X}_\pi$  contains at least the centered Bernoulli variables.

Note that this assumption is related to the Fréchet-derivability of  $\Pi$  in the direction of the space  $\text{Vect}_{\mathbb{N}}(H_{X_1}, \dots, H_{X_k})$  (which denotes the set of linear combinations of the  $H_{X_i}$  with coefficients in  $\mathbb{N}$ ) at the point  $H_{Z_1}$ , and ensures the linearity and continuity of the tangent function (derivative at  $H_{Z_1}$ ). The formulation with  $\text{Vect}_{\mathbb{N}}(H_{X_1}, \dots, H_{X_k})$  is introduced in order to cope with variables  $X_i$  which are not divisible. We put  $M_\eta \max(1, \|H_Y\|_{K,p}, \|H_{Y'}\|_{K,p})$  instead of  $M_\eta$  so that this assumption concerns in fact only the neighborhood of the gaussian variable  $Z_1$ .

We will see in Appendix C that both examples of previous section satisfy this regularity assumption.

## 3 Properties of the measures

### 3.1 Elementary properties

**Proposition 3.1** *If a performance measure  $\pi$  satisfies assumptions (A), then*

- (i)  $\pi$  has values in  $[0, +\infty]$ , with  $\pi(0) = 0$ .
- (ii)  $\forall \alpha \in \mathbb{R}^*, \pi(\alpha X) = \pi(X)$
- (iii) (2.1) can be generalised to  $n$  independent variables, for any  $n \in \mathbb{N}^*$ .
- (iv) If  $X, X_1, \dots, X_n$  are i.i.d., then

$$(3.1) \quad \pi(X_1 + \dots + X_n) = n\pi(X)$$

- (v) For  $X$  and  $Y$  independent,  $\pi(X + Y) \leq \pi(X) + \pi(Y)$ .

Proof: (i) Taking  $X = 0$  in (AT) (i), we know that there exists a gaussian variable  $Z$  such that  $\pi(Z) \in \mathbb{R}$ . Now, for  $Y$  independent from  $Z$ , we have  $\pi(Y) + \pi(Z) = \max_{\alpha, \beta \in \mathbb{R}} \pi(\alpha Y + \beta Z) \geq \pi(Z)$

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<sup>9</sup>We assume that  $X$  is independent of  $\{Z_n, n \in \mathbb{N}\}$ .

(take  $(\alpha, \beta) = (0, 1)$ ). Then  $\pi(Y) \geq 0$ .

(A1) with  $X = Y = 0$  yields  $\pi(0) = 2\pi(0)$ . For  $Z'$  independent of  $Z$ , with same law, we have  $0 \leq \pi(0) = \pi(0Z + 0Z') \leq 2\pi(Z) < +\infty$ . We conclude that  $\pi(0) = 0$ .

We note in particular that  $\lambda_X = 0$  implies  $\pi(X) = 0$ .

(ii) comes from  $\pi(\alpha X) = \max_{\beta} \pi(\beta \alpha X + 0) = \pi(X) + \pi(0)$  and (i).

(iii) For given independent  $X, Y, Z$  in  $\mathcal{X}$  and  $\gamma, \delta$  in  $\mathbb{R}$ , we have:

$$\begin{aligned} \pi(X) + \pi(\gamma Y + \delta Z) &= \max_{\alpha, \beta} \pi(\alpha X + \beta(\gamma Y + \delta Z)) \text{ then} \\ \pi(X) + \pi(Y) + \pi(Z) &= \max_{\gamma, \delta} [\pi(X) + \pi(\gamma Y + \delta Z)] = \max_{\alpha, \lambda, \mu} \pi(\alpha X + \lambda Y + \mu Z). \end{aligned}$$

The property for  $n$  variables is then obtained by induction.

(iv) If  $\lambda_X = 0$ , (3.1) comes from  $0 \leq \pi(X_1 + \dots + X_n) \leq \max_{\alpha_1, \dots, \alpha_n} \pi(\alpha_1 X_1 + \dots + \alpha_n X_n) = \pi(X_1) + \dots + \pi(X_n) = n\pi(X) = 0$ .

Else, from (ii), we have  $\pi(X_1 + \dots + X_n) = \pi(\lambda_X(X_1 + \dots + X_n)) = \max_{\alpha_1, \dots, \alpha_n} \pi(\alpha_1 X_1 + \dots + \alpha_n X_n)$ , since  $\lambda_{X_i} = \lambda_X$ . Thus  $\pi(X_1 + \dots + X_n) = n\pi(X)$ .

This property is of particular interest for infinitely divisible laws as will be seen for the gaussian case below.

(v) Consider  $\alpha = \beta = 1$  in the function maximized in (A1).

This property means that  $\pi$  is sub-additive<sup>10</sup> for independent trades, while it is additive when the trades are in the right proportions: if  $\lambda_X, \lambda_Y \in \mathbb{R}$ , we have  $\pi(\lambda_X X + \lambda_Y Y) = \pi(X) + \pi(Y)$ .

## 3.2 Gaussian variables

**Theorem 3.2** *Under assumptions (A), a performance measure  $\pi$  is proportional to the squared Sharpe ratio on the set of gaussian variables.*

Proof: we assume, for example, that  $x \mapsto \pi(\mathcal{N}(x, x))$  is bounded in a neighborhood of 1:

$$\exists \eta, M > 0 \text{ such that } |x - 1| < \eta \Rightarrow \pi(\mathcal{N}(x, x)) \leq M.$$

We set, for  $x \geq 0$ ,  $f(x) = \pi(\mathcal{N}(x, x))$ . We have  $f(0) = 0$ . For  $p, q$  in  $\mathbb{N}^*$  and  $x$  in  $\mathbb{R}$ ,  $\mathcal{N}(px, px)$  is the law of the sum of  $p$  i.i.d. variables with law  $\mathcal{N}(x, x)$  and of the sum of  $q$  i.i.d. variables with law  $\mathcal{N}(\frac{p}{q}x, \frac{p}{q}x)$ . Then, from (3.1),  $pf(x) = qf(\frac{p}{q}x)$  and we get:

$$\forall r \in \mathbb{Q}^+, f(rx) = rf(x)$$

For  $x, y \geq 0$ , the sum of two independent variables with law  $\mathcal{N}(x, x)$  and  $\mathcal{N}(y, y)$  has law  $\mathcal{N}(x + y, x + y)$ , then, from proposition 3.1 (v), we have  $f(x + y) \leq f(x) + f(y)$ .

We set, for  $x > 0$ ,  $g(x) = \frac{f(x)}{x}$ . For any  $x > 0$ , we have  $\forall r \in \mathbb{Q}^+, g(rx) = \frac{rf(x)}{rx} = g(x)$ . But  $\exists r \in \mathbb{Q}^+$  such that  $rx \in ]1 - \eta, 1 + \eta]$ , consequently  $g$  is bounded by  $M$  on  $\mathbb{R}^+$ . Last, for  $x, y > 0$ , we have  $g(x + y) \leq \frac{f(x) + f(y)}{x + y} = \frac{x}{x + y}g(x) + \frac{y}{x + y}g(y)$ . Let us prove that those three

<sup>10</sup>For coherent risk measures, the sub-additivity reflects the benefits of diversification of a portfolio. Here the interpretation is different.  $\pi(X) + \pi(Y)$  corresponds to the best performance achieved with  $X$  and  $Y$  by varying their relative quantities. In  $X + Y$ , the proportion of each opportunity may not be optimal.

properties imply that  $g$  is constant on  $\mathbb{R}^{*+}$ .

For  $x > 0$ , we consider  $(r_n)$  in  $\mathbb{Q}^+$ , decreasing and converging to  $\frac{1}{x}$ . We have  $r_n x = 1 + (r_n x - 1)$ , then  $g(x) = g(r_n x) \leq \frac{1}{r_n x} g(1) + \frac{r_n x - 1}{r_n x} g(r_n x - 1) \leq \frac{1}{r_n x} g(1) + \frac{r_n x - 1}{r_n x} M$ . Letting  $n$  go to  $+\infty$ , we get  $g(x) \leq g(1)$ .

Now we consider  $(r'_n)$  in  $\mathbb{Q}^+$ , increasing and converging to  $\frac{1}{x}$ . From  $1 = r'_n x + (1 - r'_n x)$ , we get  $g(1) \leq r'_n x g(r'_n x) + (1 - r'_n x) g(1 - r'_n x) \leq r'_n x g(x) + (1 - r'_n x) M$ , then  $g(1) \leq g(x)$ .

This proves that  $g(1) = g(x)$ . Thus, we have for  $x \in \mathbb{R}^+$ ,  $f(x) = x f(1)$ , and for  $m, \sigma \in \mathbb{R}^*$ :  $\pi(\mathcal{N}(m, \sigma^2)) = \pi(\frac{\sigma^2}{m} \mathcal{N}(\frac{m^2}{\sigma^2}, \frac{m^2}{\sigma^2})) = f(\frac{m^2}{\sigma^2}) = \frac{m^2}{\sigma^2} f(1)$ . In conclusion, for any gaussian  $X$ , we have  $\pi(X) = Sh^2(X) \pi(\mathcal{N}(1, 1))$ , which proves theorem 3.2.  $\square$

A performance measure  $\pi$  is defined up to a multiplicative constant, that will be fixed by choosing  $\pi_1 = \pi(\mathcal{N}(1, 1))$ . We assume that  $\pi$  satisfies assumptions (A). Then the theorem proves that necessarily  $\pi_1 \in ]0, +\infty[$ , since (AT) excludes  $\pi \equiv 0$  or  $+\infty$  on the set of gaussian variables. Thus  $\pi$  provides the same portfolio ranking as the Sharpe ratio for the variables with a positive Sharpe ratio. This conclusion generalizes the results of Hodges [18] and Stutzer [28].

**Normalization 3.3** *We choose  $2\pi_1 = 1$ . Then for  $X$  gaussian,  $\pi(X) = \frac{1}{2} Sh^2(X)$ .*

From section 2.2.1, we then know that for  $X$  and  $Y$  gaussian and independent,  $\sup_{\alpha, \beta} \pi(\alpha X + \beta Y)$  is achieved for any  $(\alpha, \beta)$  proportional to  $(\frac{E(X)}{V(X)}, \frac{E(Y)}{V(Y)})$ . Consequently  $\lambda_X$  is proportional to  $\frac{E(X)}{V(X)}$ , with a constant independent of  $X$ . This constant is necessarily  $\lambda_1$ . Then we have for any  $\pi$  satisfying assumptions (A):

$$(3.2) \quad \boxed{\text{for } X \text{ gaussian, } \pi(X) = \frac{1}{2} Sh^2(X) = \frac{1}{2} \frac{E(X)^2}{V(X)} \text{ and } \lambda_X = \lambda_1 \frac{E(X)}{V(X)}}$$

In particular, for  $x > 0$ ,  $Z_x$  defined in notation 2.4 satisfies

$$(3.3) \quad \pi(Z_x) = x \text{ and } \lambda_{Z_x} = 1$$

## 4 Characterization of the measures

We want to characterize the measures  $\pi$  satisfying assumptions (A), and show that any of them can be expressed as

$$(4.1) \quad \pi(X) = \sup_{\lambda} \mathcal{J}(H_{\lambda X}), \quad \forall X \in \mathcal{X}$$

with  $\mathcal{J}$  linear and continuous. For any random variable  $X$ ,  $H_X$  denotes the function of definition 2.1: we have  $H_X(\lambda) = -\ln E(e^{-\lambda X})$  for  $\lambda \in D_X = \{\lambda \in \mathbb{R} | E(e^{-\lambda X}) < +\infty\}$ .

## 4.1 Properties of the log-Laplace transform

It is well known that  $D_X$  is an interval containing 0 and that  $H_X$  is a concave function on  $D_X$ , strictly concave if and only if  $X$  is non constant (see for example [19]).  $H_X$  is called *degenerate* if  $D_X = \{0\}$ <sup>11</sup>. When  $X \in \mathcal{X}$  (ie  $0 \in \overset{\circ}{D}_X$ ), the distribution of  $X$  is uniquely determined by  $H_X$  (which is said to be *proper*), and  $H_X$  is analytic in  $\overset{\circ}{D}_X$ . The derivatives of  $H_X$  at 0 exist, as do all moments of the distribution, and we have  $H_X(0) = H'_X(0) = E(X)$ ,  $H''_X(0) = -V(X)$ ,  $H'''_X(0) = E(X - E(X))^3$ , and  $H^{(4)}_X(0) = -[E(X - E(X))^4 - 3V(X)^2]$ .

For  $X \sim \mathcal{N}(m, \sigma^2)$ , we have  $D_X = \mathbb{R}$  and  $\forall \lambda \in \mathbb{R}$ ,  $H_X(\lambda) = m\lambda - \frac{\sigma^2}{2}\lambda^2$ .

If  $E(X) = 0$ , from  $H$  concave and  $H_X(0) = H'_X(0) = 0$ , we get that  $H$  is non positive. Else, we have  $H_X(\lambda) = H_{X-E(X)}(\lambda) + \lambda E(X)$ . For  $\lambda \in \mathbb{R}$ ,  $H'_X(\lambda) = \frac{E(Xe^{-\lambda X})}{E(e^{-\lambda X})}$  is the mean of  $X$  under the probability  $P_X^\lambda$  defined by its Radon-Nikodym derivative:

$$(4.2) \quad \frac{dP_X^\lambda}{dP} = \frac{e^{-\lambda X}}{E(e^{-\lambda X})}$$

while  $H''_X(\lambda) = \frac{-E(X^2 e^{-\lambda X})E(e^{-\lambda X}) + [E(X e^{-\lambda X})]^2}{[E(e^{-\lambda X})]^2}$  is the opposite of the variance of  $X$  under  $P_X^\lambda$ .

Since  $H_X \in \mathcal{H}$  is  $C^\infty$  on  $\overset{\circ}{D}_X$ , to measure the distance between two elements of  $\mathcal{H}$ , we will use the family of semi-norms of definition 2.3. For  $K$  compact,  $p \in \mathbb{N}$  and  $H_X, H_Y \in \mathcal{H}$ , we get if  $K \subset \overset{\circ}{D}_X \cap \overset{\circ}{D}_Y$ :  $\|H_X - H_Y\|_{K,p} = \max_{0 \leq k \leq p} \max_K |H_X^{(k)} - H_Y^{(k)}|$ .

**Notation 4.1** For  $K \subset \mathbb{R}$ , we set  $\mathcal{X}_K = \{X \in \mathcal{X} \mid K \subset \overset{\circ}{D}_X\}$  and  $\mathcal{H}_K = \{H_X \mid K \subset \overset{\circ}{D}_X\}$ .

Remarks: •  $\|\cdot\|_{K,p}$  is finite on  $\mathcal{H}_K$ , since any  $H_X$  is  $C^\infty$  on  $\overset{\circ}{D}_X$ .

- If  $X \in \mathcal{X}$ ,  $\forall \lambda \in \mathbb{R}$ ,  $\lambda X \in \mathcal{X}$ , and  $\forall \lambda \neq 0$ ,  $\lambda x \in D_X \Leftrightarrow x \in D_{\lambda X}$  (i.e.  $D_{\lambda X} = \frac{1}{\lambda}D_X$ ).
- Several studies on risk or performance measures are limited to bounded variables  $X$ . Then  $D_X = \mathbb{R}$ , which simplifies some considerations.

**Proposition 4.2** For  $K$  compact,  $p \in \mathbb{N}$  and  $X \in \mathcal{X}$ ,  $\lambda \mapsto H_{\lambda X}$  is continuous for  $\|\cdot\|_{K,p}$  on the non empty open set  $\{\lambda \in \mathbb{R} \mid \lambda X \in \mathcal{X}_K\}$ .

Proof: for  $X \in \mathcal{X}$  and  $K$  compact,  $\{\lambda \in \mathbb{R} \mid \lambda X \in \mathcal{X}_K\} = \{\lambda \in \mathbb{R} \mid \lambda K \subset \overset{\circ}{D}_X\}$  is an open interval containing 0.

We have, if  $x, \lambda x \in \overset{\circ}{D}_X$ ,  $H_{\lambda X}(x) = H_X(\lambda x)$  then  $H_{\lambda X}^{(k)}(x) = \lambda^k H_X^{(k)}(\lambda x)$ . Let us assume  $\lambda_0 X \in \mathcal{X}_K$ , then for  $|\lambda - \lambda_0|$  small enough,  $\lambda K \subset \overset{\circ}{D}_X$ . For  $0 \leq k \leq p$ ,  $H_X^{(k)}$  is uniformly continuous on a neighborhood of  $K$ , therefore  $\|H_{\lambda X} - H_{\lambda_0 X}\|_{K,p} = \max_{\substack{x \in K \\ 0 \leq k \leq p}} |\lambda^k H_X^{(k)}(\lambda x) - \lambda_0^k H_X^{(k)}(\lambda_0 x)|$  goes to 0 when  $|\lambda - \lambda_0| \rightarrow 0$ .

<sup>11</sup>For example the Cauchy distribution (more generally, any distribution with both tails behaving as negative powers) has a degenerate  $H_X$ .

## 4.2 Examples

We prove that both examples of section 2.2, the squared Sharpe ratio and Hodges' measure, can be written as in (4.1).

- For a given  $X \in \mathcal{X}^* = \mathcal{X} \setminus \{0\}$ , the function  $\lambda \mapsto E(X)\lambda - V(X)\frac{\lambda^2}{2}$  has its maximum at  $\frac{E(X)}{V(X)}$ , where it is worth  $\frac{E(X)^2}{2V(X)} = \frac{1}{2}Sh^2(X)$ , including the case  $Sh^2(X) = +\infty$ . We have therefore  $\frac{1}{2}Sh^2(X) = \sup_{\lambda \in \mathbb{R}} [E(X)\lambda - \frac{V(X)}{2}\lambda^2] = \sup_{\lambda \in \mathbb{R}} [H'_{\lambda X}(0) + \frac{1}{2}H''_{\lambda X}(0)]$ , i.e.

$$\boxed{\frac{1}{2}Sh^2(X) = \sup_{\lambda \in \mathbb{R}} \mathcal{J}(H_{\lambda X}) \text{ with } \mathcal{J}(H) = H'(0) + \frac{1}{2}H''(0)}$$

$\mathcal{J}$  is linear and continuous for  $\|\cdot\|_{K,2}$  with  $K = \{0\}$ . Note that  $\mathcal{J}$  is not unique since any function  $H \mapsto \alpha H'(0) + \frac{\alpha^2}{2}H''(0)$  can be used, for  $\alpha \in \mathbb{R}^*$ , with a maximum achieved at  $\frac{E(X)}{\alpha V(X)}$ .

- We have  $Hod(X) = \sup_{\lambda \in \mathbb{R}} H_X(\lambda)$  i.e.

$$\boxed{Hod(X) = \sup_{\lambda \in \mathbb{R}} \mathcal{J}(H_{\lambda X}) \text{ with } \mathcal{J}(H) = H(1)}$$

$\mathcal{J}$  is linear and continuous for  $\|\cdot\|_{K,0}$  with  $K = \{1\}$ . Any function  $H \mapsto H(\alpha)$ , with  $\alpha \in \mathbb{R}^*$ , can be used.

## 4.3 Construction of $\mathcal{J}$ , additive on $\mathcal{H}_\pi$

In this section, for a performance measure  $\pi$  satisfying the assumptions of section 2, we show how to build a function  $\mathcal{J}$  achieving representation (4.1). Assumptions (A) are supposed to hold throughout section 4.3. Assumption (R) will be added from section 4.3.3 onward.

### 4.3.1 Definition of $J(X)$

**Lemma 4.3** For  $\pi : \mathcal{X} \rightarrow \overline{\mathbb{R}}$  satisfying assumptions (A) and  $X \in \mathcal{X}$ ,

- (i) the function  $x \mapsto \pi(Z_x + X) - \pi(Z_x)$  is non increasing on  $\mathbb{R}^+$ ,
- (ii)  $\pi(Z_x + X)$  is finite for  $x$  large enough.

Proof: (i) For  $x, h \geq 0$ ,  $Z_{x+h}$  can be written  $Z_x^1 + Z_h^2$  with  $Z_x^1$  and  $Z_h^2$  independent,  $Z_x^1 \sim \mathcal{N}(2x\lambda_1, 2x\lambda_1^2)$  and  $Z_h^2 \sim \mathcal{N}(2h\lambda_1, 2h\lambda_1^2)$ . Then we have, from sub-additivity and (3.3):

$$\pi(Z_{x+h} + X) - \pi(Z_{x+h}) \leq \pi(Z_x^1 + X) + \pi(Z_h^2) - \pi(Z_{x+h}) = \pi(Z_x + X) - \pi(Z_x).$$

(ii) From assumption (AT), there exists  $Z$  gaussian such that  $\pi(Z + X) \in \mathbb{R}$ . Then, for  $Y \sim \mathcal{N}(\frac{2}{\lambda_1}V(Z) - E(Z), V(Z))$  independent of  $Z$ , we have  $Y + Z \sim \mathcal{N}(\frac{2}{\lambda_1}V(Z), 2V(Z))$ , which is the law of  $Z_{x_0}$  for  $x_0 = V(\frac{Z}{\lambda_1})$ . Then  $\pi(Z_{x_0} + X) = \pi(Y + Z + X) \leq \pi(Y) + \pi(Z + X) \in \mathbb{R}$ , from (3.2) and  $\pi \geq 0$ . We then use (3.3) and (i) to get  $\pi(Z_x + X) < +\infty$  for  $x \geq x_0$ .

**Definition 4.4** For  $X \in \mathcal{X}$ , we set

$$\boxed{J(X) = \lim_{x \rightarrow +\infty} \searrow [\pi(Z_x + X) - \pi(Z_x)]}$$

The function  $J$  on  $\mathcal{X}$  defines a function on  $\mathcal{H}$ : for  $H_X \in \mathcal{H}$ , we set  $\mathcal{J}(H_X) = J(X)$ .

From previous lemma,  $J$  has values in  $\mathbb{R} \cup \{-\infty\}$  and can be computed with  $x$  varying in  $\mathbb{N}$ . Moreover the set  $\mathcal{X}_\pi$  of notation 2.5 is the set of variables  $X \in \mathcal{X}$  such that  $J(\lambda X) > -\infty$  for  $\lambda$  in a neighborhood of 1.

Interpretation in terms of a derivative of  $\Pi$ :

If  $X$  is infinitely divisible,  $\mathcal{J}(H_X)$  can be interpreted in terms of a Fréchet-derivative of  $\Pi$ :

For a given  $n$ , there is a random variable  $\xi_n$  such that  $X$  has the same law as the sum of  $n$  i.i.d. variables with the law of  $\xi_n$ . Then  $\pi(Z_n + X) = n\pi(Z_1 + \xi_n)$  and

$$\mathcal{J}(H_X) = \lim_{n \rightarrow +\infty} [n\pi(Z_1 + \xi_n) - \pi(Z_n)] = \lim_{n \rightarrow +\infty} \frac{\pi(Z_1 + \xi_n) - \pi(Z_1)}{\frac{1}{n}}, \text{ from (3.3).}$$

From (2.3), we have  $H_{Z_1 + \xi_n} = H_{Z_1} + H_{\xi_n} = H_{Z_1} + \frac{1}{n}H_X$ , then:

$$(4.3) \quad \mathcal{J}(H_X) = \lim_{n \rightarrow +\infty} \frac{\Pi(H_{Z_1} + \frac{1}{n}H_X) - \Pi(H_{Z_1})}{\frac{1}{n}}$$

i.e.  $\mathcal{J}(H_X)$  corresponds to the derivative of  $\Pi$  in the direction  $H_X$  at the point  $H_{Z_1}$ .

**Proposition 4.5** *Under assumptions (A),  $\mathcal{J}$  satisfies representation formula (4.1). For any optimal weight function  $X \mapsto \lambda_X$ , we have:*

$$(4.4) \quad \boxed{\forall X \in \mathcal{X}, \quad \pi(X) = \sup_{\lambda \in \mathbb{R}} J(\lambda X) = \begin{cases} J(\lambda_X X) & \text{if } \lambda_X \in \mathbb{R} \\ \lim_{\lambda \rightarrow \lambda_X} J(\lambda X) & \text{if } |\lambda_X| = +\infty \end{cases}}$$

and for  $X, Y \in \mathcal{X}$  independent, such that  $\lambda_X, \lambda_Y \in \mathbb{R}$ :

$$(4.5) \quad J(\lambda_X X + \lambda_Y Y) = J(\lambda_X X) + J(\lambda_Y Y)$$

In (4.4),  $J$  appears itself as a *performance measure*, and  $\pi(X)$  corresponds to the measure obtained by holding  $X$  in the optimal quantity. Since  $J(0) = 0$ , when  $J(\lambda X) < 0$ , the investor prefers (according to measure  $J$ ) to hold no asset  $X$  than a quantity  $\lambda$  of it. The second result will be extended in section 4.3.3, where we address the additivity of  $J$ .

Proof: we consider  $X \in \mathcal{X}$ . We have for  $\lambda \in \mathbb{R}$ ,  $\forall x \geq 0$ ,  $\pi(Z_x + \lambda X) \leq \pi(Z_x) + \pi(X)$ , thus  $J(\lambda X) \leq \pi(X)$  and

$$(4.6) \quad \sup_{\lambda \in \mathbb{R}} J(\lambda X) \leq \pi(X)$$

If  $\lambda_X \in \mathbb{R}$ , from  $\lambda_{Z_x} = 1$  we have  $\pi(Z_x + \lambda_X X) = \sup_{\alpha, \beta} \pi(\alpha Z_x + \beta X) = \pi(Z_x) + \pi(X)$ ,

then  $J(\lambda_X X) = \lim_{x \rightarrow +\infty} [\pi(Z_x + \lambda_X X) - \pi(Z_x)] = \pi(X)$ , which proves equality in (4.6).

If  $|\lambda_X| = +\infty$ ,  $\sup_{\lambda \in \mathbb{R}} J(\lambda X) > \pi(X)$  would imply the existence of  $\varepsilon > 0$  such that, for  $\lambda$  large enough,  $J(\lambda X) > \pi(X) + \varepsilon$ . Then for  $x \geq 0$ ,  $\pi(Z_x + \lambda X) - \pi(Z_x) \geq J(\lambda X) > \pi(X) + \varepsilon$ , but it would contradict  $\lim_{\lambda \rightarrow \lambda_X} \pi(Z_x + \lambda X) = \pi(Z_x) + \pi(X)$ . Then equality in (4.6) hold again.

The second result follows from  $\lambda_{\lambda_X X + \lambda_Y Y} = 1$  which can be proved using proposition 3.1 (iii): we have for  $Z \in \mathcal{X}$ ,  $\sup_{\alpha, \beta} \pi(\alpha(\lambda_X X + \lambda_Y Y) + \beta Z) \leq \pi(X) + \pi(Y) + \pi(Z)$  and this upper bound is achieved for  $(\alpha, \beta) = (1, \lambda_Z)$ .

**Notation 4.6** We denote by  $J_1$  the function  $J$  corresponding to  $\lambda_1 = 1$ :

$$J_1(X) = \lim_{x \rightarrow +\infty} [\pi(N_x + X) - x] \text{ with } N_x \sim \mathcal{N}(2x, 2x).$$

We have

$$(4.7) \quad \forall X \in \mathcal{X}, \quad J(X) = J_1\left(\frac{X}{\lambda_1}\right)$$

Therefore, according to (4.4), the value of  $\lambda_1$  appears as a *scale factor*. If  $X \mapsto \lambda_X$  is an optimal weight function for  $J_1$ , then  $X \mapsto \lambda_1 \lambda_X$  is an optimal weight function for  $J$ . Note that changing the value of  $\lambda_1$  does not affect  $\pi$ .

### 4.3.2 Elementary properties of $J$

We prove easily the following properties for  $J$ :

**Proposition 4.7** Under assumptions (A), we have,  $\forall X, Y \in \mathcal{X}$ :

- (i) For  $X$  gaussian,  $J(X) = E(\frac{X}{\lambda_1}) - \frac{1}{2}V(\frac{X}{\lambda_1})$ . Thus  $\mathcal{X}_\pi$  contains all gaussian variables.
- (ii) For  $X$  and  $Y$  independent,  $J(X + Y) \leq J(X) + J(Y)$ .
- (iii)  $\forall m \in \mathbb{R}$ ,  $J(X + m) = J(X) + \frac{m}{\lambda_1}$ .
- (iv) For  $X$  and  $Y$  independent, with  $X$  gaussian,

$$(4.8) \quad J(X + Y) = J(X) + J(Y)$$

Proof: it is sufficient to prove the proposition for  $\lambda_1 = 1$  (we keep the notations  $J$ ,  $Z_x$ ).

(i) If  $X \sim \mathcal{N}(m, \sigma^2)$  with  $\sigma \geq 0$ , from (3.2), we have  $J(X) = \lim_{x \rightarrow +\infty} [\frac{1}{2} \frac{(2x+m)^2}{2x+\sigma^2} - x] = m - \frac{\sigma^2}{2}$ .

(ii) Let us consider  $X$  and  $Y$  independent. For  $x \geq 0$ ,  $Z_{2x}$  can be written  $Z_x^1 + Z_x^2$  where  $Z_x^1$  and  $Z_x^2$  have same law than  $Z_x$  and  $X, Y, Z_x^1, Z_x^2$  are independent. We have  $J(X + Y) = \lim_{x \rightarrow +\infty} [\pi(Z_{2x} + X + Y) - 2x] \leq \lim_{x \rightarrow +\infty} [\pi(Z_x^1 + X) + \pi(Z_x^2 + Y) - 2x]$ .

Thus  $J(X + Y) \leq \lim_{x \rightarrow +\infty} [\pi(Z_x^1 + X) - x] + \lim_{x \rightarrow +\infty} [\pi(Z_x^2 + Y) - x] = J(X) + J(Y)$ .

(iii) For any constant  $m$ , we get  $J(X) = J(X + m - m) \leq J(X + m) - m$  and  $J(X + m) \leq J(X) + m$ , then  $J(X + m) = J(X) + m$ .

(iv) Assume  $X \sim \mathcal{N}(m, \sigma^2)$ , then  $Z_x + X \sim \mathcal{N}(2x + \sigma^2 + (m - \sigma^2), 2x + \sigma^2)$ , thus  $Z_x + X$  and  $Z_{x+\frac{\sigma^2}{2}} + m - \sigma^2$  have same law, while  $J(X) = m - \frac{\sigma^2}{2}$ .

$$\begin{aligned} \text{Then } J(X + Y) &= \lim_{x \rightarrow +\infty} [\pi(Z_x + X + Y) - x] \\ &= \lim_{x \rightarrow +\infty} [\pi(Z_{x+\frac{\sigma^2}{2}} + Y + m - \sigma^2) - (x + \frac{\sigma^2}{2})] + \frac{\sigma^2}{2} \\ &= \lim_{x \rightarrow +\infty} [\pi(Z_x + Y + m - \sigma^2) - x] + \frac{\sigma^2}{2} = J(Y + m - \sigma^2) + \frac{\sigma^2}{2} \\ &= J(Y) + m - \sigma^2 + \frac{\sigma^2}{2} = J(Y) + J(X). \end{aligned}$$



### 4.3.3 Additivity of $J$ , under additional regularity assumption

In order to obtain the additivity<sup>12</sup> of  $J$  on  $\mathcal{X}_\pi$ , we need a continuity assumption on  $\pi$ , to get  $\lambda \mapsto J(\lambda X)$  continuous at  $\lambda = 1$  (see proof of proposition 4.10). However, to further extend  $\mathcal{J}$  to  $Vect(\mathcal{H}_\pi)$  in the next section, we need the stronger (and more complex) assumption (R) given in section 2.3, which implies more regularity for  $\mathcal{J}$ . Using notation 4.1, we recall assumption (R):  $\mathcal{X}_\pi$  contains the centered Bernoulli variables and

$$\begin{aligned} \exists p \in \mathbb{N}, K \text{ compact such that } \mathcal{X}_\pi \subset \mathcal{X}_K \text{ and } M > 0 \text{ such that } \forall X_1, \dots, X_k \in \mathcal{X}_\pi, \\ \exists \eta_0 > 0, \forall \eta \leq \eta_0, \exists M_\eta > 0 \text{ such that } \forall H_Y, H_{Y'} \in Vect_{\mathbb{N}}(H_{X_1}, \dots, H_{X_k}), \\ n \geq M_\eta \max(1, \|H_Y\|_{K,p}, \|H_{Y'}\|_{K,p}) \Rightarrow |\pi(Z_n + Y) - \pi(Z_n + Y')| \leq M \|H_Y - H_{Y'}\|_{K^\eta,p}. \end{aligned}$$

**Proposition 4.8** *Under assumptions (A) and (R),  $\mathcal{J}$  is finite and uniformly Lipschitz on  $\mathcal{H}_\pi = \{H_X \mid X \in \mathcal{X}_\pi\}$ , for the semi-norm  $\|\cdot\|_{K,p}$ , with  $(K,p)$  as in (R).*

Proof: we take  $(K,p,M)$  as in (R). Then the semi-norm  $\|\cdot\|_{K,p}$  is finite on  $\mathcal{H}_\pi$  (since  $\mathcal{X}_\pi \subset \mathcal{X}_K$ ). For  $X_1, X_2 \in \mathcal{X}_\pi$ , (R) with  $k = 2$  implies:  $\exists \eta_0 > 0, \forall \eta \leq \eta_0, \exists M_\eta > 0$  such that  $n \geq M_\eta \Rightarrow |\pi(Z_n + X_1) - \pi(Z_n + X_2)| \leq M \|H_{X_1} - H_{X_2}\|_{K^\eta,p}$ .

For any  $\eta \leq \eta_0$ , taking  $n \rightarrow +\infty$ , we get  $|J(X_1) - J(X_2)| \leq M \|H_{X_1} - H_{X_2}\|_{K^\eta,p}$ . Since  $K \subset \overset{\circ}{D}_{X_i}$ , this upper bound is finite for  $\eta$  small enough. Making  $\eta \rightarrow 0$ , we prove:

$$(4.9) \quad \forall X_1, X_2 \in \mathcal{X}_\pi, \quad |\mathcal{J}(H_{X_1}) - \mathcal{J}(H_{X_2})| \leq M \|H_{X_1} - H_{X_2}\|_{K,p}$$

i.e.  $\mathcal{J}$  is uniformly Lipschitz on  $\mathcal{H}_\pi$ . The finiteness of  $\mathcal{J}$  on  $\mathcal{H}_\pi$  follows, since  $0 \in \mathcal{X}_\pi$ .

In particular,  $\mathcal{J}$  is continuous on  $\mathcal{H}_\pi$  for  $\|\cdot\|_{K,p}$ . We are going to show that this property implies the additivity of  $\mathcal{J}$  on  $\mathcal{H}_\pi$ .

**Lemma 4.9** *Under assumptions (A), for  $X \in \mathcal{X}_\pi$  such that  $\pi(X) < +\infty$  and for a given optimal weight function  $X \mapsto \lambda_X$ , the set  $\Lambda_X = \{\lambda_{X+Z} \mid Z \text{ gaussian, independent from } X\}$  is dense in a neighborhood of 1.*

Proof: again it is sufficient to prove it with  $\lambda_1 = 1$ . Since  $X \in \mathcal{X}_\pi$ , we have  $J(\lambda X) > -\infty$  for  $\lambda$  in a neighborhood of 1, say  $]1 - \varepsilon, 1 + \varepsilon[$ .

We consider  $\lambda_2 < \lambda_3$  in  $]1 - \varepsilon, 1 + \varepsilon[$  and show that exists  $\lambda \in \Lambda_X$  between  $\lambda_2$  and  $\lambda_3$ . We set  $\lambda^* = \frac{\lambda_2 + \lambda_3}{2}$  and consider  $Z \sim \mathcal{N}(\lambda^* \sigma^2, \sigma^2)$  independent from  $X$ . We have from propositions 4.5 and 4.7:

$$\pi(X + Z) = \sup_{\lambda} \left[ J(\lambda X) + \lambda \lambda^* \sigma^2 - \frac{\lambda^2 \sigma^2}{2} \right] = \sup_{\lambda} f(\lambda) + \frac{(\lambda^* \sigma)^2}{2}$$

with  $f(\lambda) = J(\lambda X) - \frac{\sigma^2}{2}(\lambda - \lambda^*)^2$ . From assumption (A2),  $\sup_{\lambda} f(\lambda)$  is then achieved at  $\lambda_{X+Z}$ . We have for  $\lambda \notin ]\lambda_2, \lambda_3[$ ,  $f(\lambda) \leq \sup_{\lambda'} J(\lambda' X) - \frac{\sigma^2}{2}(\lambda_3 - \lambda^*)^2 = \pi(X) - \frac{\sigma^2}{2}(\lambda_3 - \lambda^*)^2$ .

<sup>12</sup>Such a property appears already in [16] where a premium calculation principle is called *additive*, if the premium assigned to the sum of two independent risks is the sum of the premiums that are assigned to the two risks individually.

This upper bound converges to  $-\infty$  when  $\sigma$  converges to  $+\infty$  (from  $\pi(X) < \infty$ ), while  $f(\lambda^*) = J(\lambda^*X)$  is finite and independent of  $\sigma$ . Then for large  $\sigma$ ,  $\sup f$  is achieved on  $]\lambda_2, \lambda_3[$ . We get  $\lambda_{X+Z} \in ]\lambda_2, \lambda_3[$  and  $\Lambda_X \cap ]\lambda_2, \lambda_3[ \neq \emptyset$ , which proves the density in  $]1-\varepsilon, 1+\varepsilon[$ .

**Proposition 4.10** *Under assumptions (A) and (R), the function  $J$  is **additive** on  $\mathcal{X}_\pi$ , i.e. (4.8) holds for any  $X$  and  $Y$  independent in  $\mathcal{X}_\pi$ . In particular  $\mathcal{X}_\pi$  and  $\mathcal{H}_\pi$  are stable under addition.*

Proof: We consider  $X, Y$  independent in  $\mathcal{X}_\pi$  and prove  $J(X + Y) = J(X) + J(Y)$ .

- We can assume  $\pi(X)$  and  $\pi(Y)$  finite. Indeed  $\pi(Z_n + X)$  and  $\pi(Z_n + Y)$  are finite for  $n$  large enough (lemma 4.3), and if we prove  $J(X + Z + Y + Z') = J(X + Z) + J(Y + Z')$  with  $Z$  and  $Z'$  gaussian and  $X, Y, Z, Z'$  independent, then (4.8) yields  $J(X + Y) = J(X) + J(Y)$ .
- Let  $(\lambda, \lambda') = (\lambda_{X+Z}, \lambda_{Y+Z'})$  in  $\Lambda_X \times \Lambda_Y$ . From lemma 4.9,  $(\lambda, \lambda')$  can be chosen arbitrarily close to  $(1, 1)$ . We have  $J(\lambda X + \lambda' Y) = J(\lambda(X + Z) + \lambda'(Y + Z')) - J(\lambda Z + \lambda' Z')$  since  $\lambda Z + \lambda' Z'$  is gaussian. By (4.5), we have  $J(\lambda(X + Z) + \lambda'(Y + Z')) = J(\lambda(X + Z)) + J(\lambda'(Y + Z'))$ , since  $\lambda, \lambda' \in \mathbb{R}$ . Using again (4.8), we get  $J(\lambda X + \lambda' Y) = J(\lambda X) + J(\lambda' Y)$ .

For  $\lambda$  and  $\lambda'$  close enough to 1, we have  $\lambda X, \lambda' Y \in \mathcal{X}_K$ . Then propositions 4.2 and 4.8 prove that  $(\lambda, \lambda') \mapsto \mathcal{J}(H_{\lambda X} + H_{\lambda' Y}) = J(\lambda X + \lambda' Y)$  is continuous at  $(1, 1)$ . Taking  $(\lambda, \lambda') \rightarrow (1, 1)$ , we get  $J(X + Y) = J(X) + J(Y)$ .

We deduce that  $\mathcal{J}$  is additive on  $\mathcal{H}_\pi$ : for  $X$  and  $Y$  independent in  $\mathcal{X}_\pi$ , we have

$$(4.10) \quad \mathcal{J}(H_X + H_Y) = J(X + Y) = J(H_X) + J(H_Y)$$

#### 4.4 Extension of $\mathcal{J}$ into a linear application on $Vect(\mathcal{H}_\pi)$ , under assumptions (A) and (R)

Under assumptions (A) and (R),  $\mathcal{J}$  is defined and additive on  $\mathcal{H}_\pi$ . In this section, with the same assumptions, we extend its definition to  $Vect(\mathcal{H}_\pi)$ , the vector space generated by  $\mathcal{H}_\pi$  and in section 5, to a larger set of  $C^\infty$  functions.

*Step 1. We extend  $\mathcal{J}$  uniquely to  $Vect_{\mathbb{Q}}(\mathcal{H}_\pi)$ , the vector space generated by  $\mathcal{H}_\pi$  with coefficients in  $\mathbb{Q}$ , as a linear application for the structure of  $\mathbb{Q}$ -vector space.*

First, the property of additivity can be generalised to the following one: for any  $H_X, H_Y \in \mathcal{H}_\pi$  and  $n, p \in \mathbb{N}$ , we have

$$(4.11) \quad \mathcal{J}(nH_X + pH_Y) = n\mathcal{J}(H_X) + p\mathcal{J}(H_Y)$$

Indeed we have  $nH_X + pH_Y = H_{X_1 + \dots + X_n + Y_1 + \dots + Y_p}$  where  $X_1, \dots, X_n, Y_1, \dots, Y_p$  are independent and  $(X_i, Y_i)$  has the same law as  $(X, Y)$ . Then  $\mathcal{J}(nH_X + pH_Y) = J(X_1 + \dots + X_n + Y_1 + \dots + Y_p) = nJ(X) + pJ(Y)$  from proposition 4.10. Note that in particular,  $Vect_{\mathbb{N}}(\mathcal{H}_\pi) = \mathcal{H}_\pi$ .

Then we note that the additivity of the log-Laplace for independent variables implies

$$(4.12) \quad Vect_{\mathbb{Q}}(\mathcal{H}_\pi) = \left\{ \frac{1}{q}H_Y - \frac{1}{q'}H_{Y'} \mid H_Y, H_{Y'} \in Vect_{\mathbb{N}}(\mathcal{H}_\pi), q, q' \in \mathbb{N}^* \right\}$$

Indeed, any element of  $Vect_{\mathbb{Q}}(\mathcal{H}_\pi)$  is the difference of two terms  $\sum_i \frac{p_i}{q_i} H_{X_i}$  with  $p_i, q_i \in \mathbb{N}^*$ ,  $X_i \in \mathcal{X}_\pi$ , and we have  $\sum \frac{p_i}{q_i} H_{X_i} = \frac{1}{\prod q_i} H_Y$  with  $H_Y \in Vect_{\mathbb{N}}(\mathcal{H}_\pi)$ .

Now, for  $H_Y, H_{Y'} \in Vect_{\mathbb{N}}(\mathcal{H}_\pi)$  and  $q, q' \in \mathbb{N}^*$ , let us set:

$$\mathcal{J} \left( \frac{1}{q}H_Y - \frac{1}{q'}H_{Y'} \right) = \frac{1}{q}\mathcal{J}(H_Y) - \frac{1}{q'}\mathcal{J}(H_{Y'})$$

This definition is consistent: if  $\frac{1}{q}H_Y - \frac{1}{q'}H_{Y'} = \frac{1}{p}H_X - \frac{1}{p'}H_{X'}$ , we get  $q'pp'H_Y + qq'p'H_{X'} = qq'p'H_X + qpp'H_{Y'}$ , therefore  $\mathcal{J}(q'pp'H_Y + qq'p'H_{X'}) = \mathcal{J}(qq'p'H_X + qpp'H_{Y'})$ . Then, using (4.11), we get  $\frac{1}{q}\mathcal{J}(H_Y) - \frac{1}{q'}\mathcal{J}(H_{Y'}) = \frac{1}{p}\mathcal{J}(H_X) - \frac{1}{p'}\mathcal{J}(H_{X'})$ .

Note that for  $q = 1$  and  $Y' = 0$ ,  $\frac{1}{q}H_Y - \frac{1}{q'}H_{Y'} = H_Y$ , which proves that the definition of  $\mathcal{J}$  on  $Vect_{\mathbb{Q}}(\mathcal{H}_\pi)$  is consistent with its definition on  $\mathcal{H}_\pi$ .

It is then easy to check that  $\mathcal{J}$  is linear (for coefficients in  $\mathbb{Q}$ ) on  $Vect_{\mathbb{Q}}(\mathcal{H}_\pi)$ .

Step 2. We prove that  $\mathcal{J}$  is continuous on  $Vect_{\mathbb{Q}}(\mathcal{H}_\pi)$  for  $\|\cdot\|_{K,p}$ , with a norm  $\|\mathcal{J}\| \leq M$ .

Since  $\mathcal{J}$  is linear, we have to prove  $|\mathcal{J}(H)| \leq M\|H\|_{K,p}$  for  $H \in Vect_{\mathbb{Q}}(\mathcal{H}_\pi)$ . From (4.12), it is sufficient to prove that for any  $H_Y, H_{Y'} \in Vect_{\mathbb{N}}(\mathcal{H}_\pi)$  and  $q, q' \in \mathbb{N}^*$ , we have:

$$(4.13) \quad \left| \mathcal{J} \left( \frac{1}{q}H_Y \right) - \mathcal{J} \left( \frac{1}{q'}H_{Y'} \right) \right| \leq M \left\| \frac{1}{q}H_Y - \frac{1}{q'}H_{Y'} \right\|_{K,p}$$

Note that (4.13) is easy to prove for  $Y$  and  $Y'$  divisible. Indeed if we can write  $Y = \xi_1 + \dots + \xi_q$ , then  $\frac{1}{q}H_Y = H_{\xi_q}$ . With  $Y' = \xi'_1 + \dots + \xi'_{q'}$ , we get from (4.9):

$$|\mathcal{J}(\frac{1}{q}H_Y) - \mathcal{J}(\frac{1}{q'}H_{Y'})| \leq M\|H_{\xi_1} - H_{\xi'_1}\|_{K,p} = M\|\frac{1}{q}H_Y - \frac{1}{q'}H_{Y'}\|_{K,p}.$$

We want to prove (4.13) with  $Y, Y'$  non divisible. Note that for  $X \in \mathcal{X}$ , we have  $\mathcal{J}(H_X) = \lim_{n \rightarrow +\infty} [\pi(Z_{nq} + X) - nq]$  then

$$(4.14) \quad \mathcal{J} \left( \frac{1}{q}H_X \right) = \lim_{n \rightarrow +\infty} \left[ \frac{1}{q}\pi(Z_{nq} + X) - n \right]$$

We have  $H_Y, H_{Y'} \in Vect_{\mathbb{N}}(H_{X_1}, \dots, H_{X_k})$  with  $X_i \in \mathcal{X}_\pi$ . From (R),  $\exists \eta_0 > 0$  such that  $\forall \eta \leq \eta_0$ ,  $\exists M_\eta > 0$  such that  $\forall H_X, H_{X'} \in Vect_{\mathbb{N}}(H_{X_1}, \dots, H_{X_k})$ ,

$$n \geq M_\eta \max(1, \|H_X\|_{K,p}, \|H_{X'}\|_{K,p}) \Rightarrow |\pi(Z_n + X) - \pi(Z_n + X')| \leq M\|H_X - H_{X'}\|_{K^\eta, p}.$$

Let us consider  $Y_1, \dots, Y_{q'}$  i.i.d. with same law as  $Y$  and  $Y'_1, \dots, Y'_q$  i.i.d. with same law as  $Y'$ . Then  $H_{Y_1 + \dots + Y_{q'}}, H_{Y'_1 + \dots + Y'_q} \in Vect_{\mathbb{N}}(H_{X_1}, \dots, H_{X_k})$ . For  $\eta \leq \eta_0$  and  $n \geq M_\eta \max(\frac{1}{qq'}, \frac{1}{q}\|H_Y\|_{K,p}, \frac{1}{q'}\|H_{Y'}\|_{K,p})$ , we have  $nqq' \geq M_\eta \max(1, q'\|H_Y\|_{K,p}, q\|H_{Y'}\|_{K,p})$ , then

$$|\pi(Z_{nqq'} + Y_1 + \dots + Y_{q'}) - \pi(Z_{nqq'} + Y'_1 + \dots + Y'_{q'})| \leq M \|q'H_Y - qH_{Y'}\|_{K^\eta, p}, \text{ and}$$

$$\left| \frac{1}{q}\pi(Z_{nq} + Y) - \frac{1}{q'}\pi(Z_{nq'} + Y') \right| \leq M \left\| \frac{1}{q}H_Y - \frac{1}{q'}H_{Y'} \right\|_{K^\eta, p}$$

From (4.14), taking  $n \rightarrow +\infty$  we deduce that:

$$(4.15) \quad \left| \mathcal{J}\left(\frac{1}{q}H_Y\right) - \mathcal{J}\left(\frac{1}{q'}H_{Y'}\right) \right| \leq M \left\| \frac{1}{q}H_Y - \frac{1}{q'}H_{Y'} \right\|_{K^\eta, p}$$

and making  $\eta \rightarrow 0$ , we get (4.13).

### Step 3. Extension to $\text{Vect}(\mathcal{H}_\pi)$

The extension of  $\mathcal{J}$  is then classical. For any  $X, Y$  independent in  $\mathcal{X}_\pi$  and  $\alpha, \beta \in \mathbb{R}$ , let us set:

$$\mathcal{J}(\alpha H_X + \beta H_Y) = \alpha \mathcal{J}(H_X) + \beta \mathcal{J}(H_Y)$$

This definition is consistent under assumption (R): let us prove that if  $\alpha H_X + \beta H_Y = \alpha' H_{X'} + \beta' H_{Y'}$ , then

$$(4.16) \quad \alpha \mathcal{J}(H_X) + \beta \mathcal{J}(H_Y) = \alpha' \mathcal{J}(H_{X'}) + \beta' \mathcal{J}(H_{Y'})$$

We assume  $\alpha, \beta, \alpha', \beta' \in \mathbb{R}^+$ . For  $\alpha, \beta, \alpha', \beta' \in \mathbb{R}$ , the same result will be obtained by rearranging the terms in  $\alpha H_X + \beta H_Y = \alpha' H_{X'} + \beta' H_{Y'}$  in order to get only non negative coefficients.

Let  $(r_{1n}, r_{2n}, r'_{1n}, r'_{2n})$  in  $(\mathbb{Q}^{++})^4$ , converging toward  $(\alpha, \beta, \alpha', \beta')$ . We write  $r_{in} = \frac{p_{in}}{q_{in}}$  with  $p_{in}, q_{in} \in \mathbb{N}^*$ . We have  $r_{1n}H_X + r_{2n}H_Y = \frac{1}{m_n}H_{W_n}$  with  $m_n = q_{1n}q_{2n}$  and  $W_n = X_1 + \dots + X_{p_{1n}q_{2n}} + Y_1 + \dots + Y_{p_{2n}q_{1n}}$  where the  $X_i, Y_i$  are independent and each  $X_i$  (resp  $Y_i$ ) has the law of  $X$  (resp  $Y$ ). Then, with obvious notations,  $r_{1n}H_X + r_{2n}H_Y - (r'_{1n}H_{X'} + r'_{2n}H_{Y'}) = \frac{1}{m_n}H_{W_n} - \frac{1}{m'_n}H_{W'_n}$  and  $H_{W_n}, H_{W'_n} \in \text{Vect}_{\mathbb{N}}(\mathcal{H}_\pi)$ .

From (4.15), we have  $|\mathcal{J}(\frac{1}{m_n}H_{W_n}) - \mathcal{J}(\frac{1}{m'_n}H_{W'_n})| \leq M \|\frac{1}{m_n}H_{W_n} - \frac{1}{m'_n}H_{W'_n}\|_{K, p}$ . We note here that we needed a uniformity in  $Y, Y'$  in assumption (R) since  $W_n$  and  $W'_n$  are changing when  $n \rightarrow +\infty$ , for  $X, Y, X', Y'$  given in  $\mathcal{X}_\pi$ .

$$\begin{aligned} \text{But } \|\frac{1}{m_n}H_{W_n} - \frac{1}{m'_n}H_{W'_n}\|_{K, p} &= \|r_{1n}H_X + r_{2n}H_Y - [r'_{1n}H_{X'} + r'_{2n}H_{Y'}]\|_{K, p} \\ &= \|(r_{1n} - \alpha)H_X + (r_{2n} - \beta)H_Y - [(r'_{1n} - \alpha')H_{X'} + (r'_{2n} - \beta')H_{Y'}]\|_{K, p} \text{ is less than} \\ &|r_{1n} - \alpha| \cdot \|H_X\|_{K, p} + |r_{2n} - \beta| \cdot \|H_Y\|_{K, p} + |r'_{1n} - \alpha'| \cdot \|H_{X'}\|_{K, p} + |r'_{2n} - \beta'| \cdot \|H_{Y'}\|_{K, p} \end{aligned}$$

and converges toward 0 when  $n \rightarrow +\infty$ .

Then  $\mathcal{J}(r_{1n}H_X + r_{2n}H_Y) - \mathcal{J}(r'_{1n}H_{X'} + r'_{2n}H_{Y'}) = \mathcal{J}(\frac{1}{m_n}H_{W_n}) - \mathcal{J}(\frac{1}{m'_n}H_{W'_n})$  goes to 0 when  $n \rightarrow +\infty$ , i.e.  $r_{1n}\mathcal{J}(H_X) + r_{2n}\mathcal{J}(H_Y) - [r'_{1n}\mathcal{J}(H_{X'}) + r'_{2n}\mathcal{J}(H_{Y'})]$  converges to 0 and this proves (4.16).

The previous proof can be easily extended to any finite number of terms, instead of two,

and we get a consistent definition of  $\mathcal{J}$  on  $Vect(\mathcal{H}_\pi)$ .

**Conclusion**  $\mathcal{J}$  is now defined on  $Vect(\mathcal{H}_\pi)$  and by construction, it is linear on this space and continuous for  $\|\cdot\|_{K,p}$ , with the same norm  $\|\mathcal{J}\| \leq M$  as in  $Vect_{\mathbb{Q}}(\mathcal{H}_\pi)$ .

Indeed, if  $H = \sum_i \alpha_i H_{X_i} \in Vect(\mathcal{H}_\pi)$ , for  $r_n^i$  converging to  $\alpha_i$ , we have  $\forall n, |\sum_i r_n^i \mathcal{J}(H_{X_i})| = |\mathcal{J}(\sum_i r_n^i H_{X_i})| \leq \|\mathcal{J}\| \cdot \|\sum_i r_n^i H_{X_i}\|_{K,p}$ . Taking  $n \rightarrow +\infty$ , we get  $|\mathcal{J}(H)| \leq \|\mathcal{J}\| \cdot \|H\|_{K,p}$ .

We have proved the following representation theorem:

**Theorem 4.11** *Under assumptions (A) and (R),  $\pi$  can be written:*

$$(4.17) \quad \forall X \in \mathcal{X}, \quad \pi(X) = \sup_{\lambda} \mathcal{J}(H_{\lambda X})$$

with  $\mathcal{J}$  linear and continuous for the semi-norm  $\|\cdot\|_{K,p}$  on  $Vect(\mathcal{H}_\pi)$ . If  $X \mapsto \lambda_X$  is an optimal weight function, the upper bound in (4.17) is achieved at  $\lambda_X$ .

|                                                                                                                                                                                                                                                                                                                                                                          |
|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <p><math>J(X)</math> is itself a <i>performance measure</i> for the investment in <math>X</math>.<br/> <math>\pi(X) = \sup_{\lambda} \mathcal{J}(H_{\lambda X})</math> is the <i>best performance</i> obtained by varying the quantity of <math>X</math>.<br/> <math>\lambda_1 = \lambda_{\mathcal{N}(1,1)}</math> is a scale factor on any optimal weight function.</p> |
|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|

## 5 Expression of $\mathcal{J}$ as a distribution

### 5.1 Definition of the distribution

We consider  $\pi$  satisfying assumptions (A) and (R) and keep the notations of last section. We work with  $\lambda_1 = 1$ , setting  $\mathcal{J}_1(H_X) = J_1(X)$  for  $X \in \mathcal{X}$ . Let us denote by  $C^p(K)$  the set of functions which are  $p$ -times derivable on the compact  $K$ . On  $Vect(\mathcal{H}_\pi) \cap C^p(K)$ , subspace of the vector space  $C^p(K)$ ,  $\mathcal{J}_1$  is a linear application with values in  $\mathbb{R}$ , which is continuous and has a norm  $\|\mathcal{J}_1\| \leq M$ . Then by the Hahn-Banach theorem,  $\mathcal{J}_1$  can be prolonged to the whole space  $C^p(K)$ , with the same norm  $\|\mathcal{J}_1\|$ , the semi-norm  $\|\cdot\|_{K,p}$  being used on  $C^p(K)$ . In particular  $\mathcal{J}_1$  is now defined on  $C_0^\infty$ , the subspace of  $C^p(K)$  of  $C^\infty$  functions with compact support. The restriction of  $\mathcal{J}_1$  to  $C_0^\infty$  is a distribution  $g$  of order  $p$  with compact support  $K_g$  included in  $K$ .

$g$  is a Schwartz distribution on  $C_0^\infty$  with the compact support  $K_g$ , therefore it can be extended as a distribution acting on the space of  $C^\infty$  functions defined on a neighborhood of  $K_g$ , space which contains  $Vect(\mathcal{H}_\pi)$  since any function of  $\mathcal{H}_\pi$  is  $C^\infty$  on  $K \supset K_g$ .

In particular, for  $X \in \mathcal{X}_\pi$ , we have:

$$(5.1) \quad \mathcal{J}_1(H_X) = \langle g, H_X \rangle$$

We construct a primitive of  $g$  with compact support

[This is a classical result, for completeness we describe the proof below.]

We denote by  $I$  the convex envelop of  $K_g \cup \{0\}$ . We consider  $\varphi_2 \in C_0^\infty$ , such that  $\varphi_2 \equiv 1$  on  $I$ . We choose  $c_1 \in \mathbb{R}$  such that  $\langle g + c_1 \delta_0, \varphi_2 \rangle = 0$  where  $\delta_0$  denotes the Dirac distribution at 0. Replacing  $g$  by  $g + c_1 \delta_0$ , we keep (5.1) for  $X \in \mathcal{X}_\pi$  since  $H_X(0) = 0$ , and we get  $\langle g, \varphi_2 \rangle = 0$ .

Let  $G$  be a primitive of the distribution  $g$  (see Schwartz [27]). We consider  $\varphi_1 \in C_0^\infty$ , with support included in  $I^c$ , the complementary set of  $I$ , and such that  $\int \varphi_1 = 1$ . Modifying  $G$  by a constant, we can assume  $\langle G, \varphi_1 \rangle = 0$ . Let us prove that the support of  $G$  is included in  $I$  and then is compact.

Let  $\varphi \in C_0^\infty$  with support included in  $I^c$ . We know (from  $\int \varphi_1 = 1$ ) that  $\varphi$  has a unique decomposition  $\varphi = \alpha_1 \varphi_1 + \psi'_1$  with  $\alpha_1 \in \mathbb{R}$  and  $\psi_1 \in C_0^\infty$ . Since  $I$  is an interval,  $\psi_1$  is constant on  $I$  then  $\psi_1 = \alpha_2 \varphi_2 + \psi_2$  with  $\psi_2 \in C_0^\infty$  and  $\psi_2 \equiv 0$  on  $I$ . We have:

$$\langle G, \varphi \rangle = \alpha_1 \langle G, \varphi_1 \rangle + \langle G, \psi'_1 \rangle = -\langle g, \psi_1 \rangle = -\alpha_2 \langle g, \varphi_2 \rangle - \langle g, \psi_2 \rangle = 0, \text{ since } \text{supp} g \subset I.$$

This proves that the support of  $G$  is included in  $I$ .

*We construct a primitive of  $2(G + C\delta_0)$ , with support included in  $I$*

By the same argument, for a good choice of  $C \in \mathbb{R}$ , we can construct a primitive  $\Gamma$  of the distribution  $2(G + C\delta_0)$  with a support included in  $I$ . From  $\Gamma' = 2(G + C\delta_0)$  we get:

$$(5.2) \quad g = G' = -C\delta'_0 + \frac{1}{2}\Gamma''$$

$G$  and  $\Gamma$  are distributions with compact support included in  $I$ , then as  $g$ , they can be extended as distributions acting on the space of  $C^\infty$  functions defined on a neighborhood of  $I$ .

If  $K'$  denotes the convex envelop of  $K \cup \{0\}$ , assumption (R) still holds with  $(K', p)$  instead of  $(K, p)$ , since  $\|\cdot\|_{K,p} \leq \|\cdot\|_{K',p}$ . Then changing  $K$  in  $K'$ , we can always assume that  $K$  is a closed interval containing 0, then containing  $I$  (note that, since  $\overset{\circ}{D}_X$  is an interval containing 0, we have  $\mathcal{X}_{K'} = \mathcal{X}_K$ ). After this change,  $\mathcal{J}_1$  is continuous for  $\|\cdot\|_{K,p}$  on  $\text{Vect}(\mathcal{H}_\pi)$ , the distributions  $g$ ,  $G$  and  $\Gamma$  have their support included in  $K$  and for  $X \in \mathcal{X}_\pi$ , we have  $J_1(X) = \langle g, H_X \rangle = -\langle G, H'_X \rangle = CH'_X(0) + \frac{1}{2}\langle \Gamma, H''_X \rangle$ .

If  $X \sim \mathcal{N}(m, \sigma^2)$ , we get  $J_1(X) = Cm - \frac{\sigma^2}{2}\langle \Gamma, 1 \rangle$ , which must be equal to  $m - \frac{\sigma^2}{2}$  from proposition 4.7. Then necessarily  $C = 1$  and  $\langle \Gamma, 1 \rangle = 1$ .

Note that for any  $c \in \mathbb{R}$ , we have  $\forall X \in \mathcal{X}_\pi$ ,  $\mathcal{J}_1(H_X) = \langle g + c\delta_0, H_X \rangle$ . In the set  $\{g + c\delta_0 \mid c \in \mathbb{R}\}$ , the distribution  $g$  satisfying (5.2) is the one such that

$$(5.3) \quad \langle g, 1 \rangle = 0$$

In conclusion, we get that  $g$ ,  $G$  and  $\Gamma$  are distributions with compact support included in  $K$  (closed interval containing 0), such that

$$(5.4) \quad g = G' = -\delta'_0 + \frac{1}{2}\Gamma'' \text{ with } \langle \Gamma, 1 \rangle = 1$$

and  $\forall X \in \mathcal{X}_\pi$ ,  $J_1(X) = \langle g, H_X \rangle = -\langle G, H'_X \rangle = E(X) + \frac{1}{2} \langle \Gamma, H''_X \rangle$ .

Example: if the distribution  $\Gamma$  is a function, we get  $\mathcal{J}_1(H_X) = H'_X(0) + \frac{1}{2} \int_{-\infty}^{+\infty} \Gamma(x) H''_X(x) dx$  with  $\int_{-\infty}^{+\infty} \Gamma(x) dx = 1$ . Note that a partial parallel can be made with the spectral measures of risk of Acerbi [1] and [2] (obtained as general combination of expected shortfalls). A “risk aversion function”  $\phi$  appears which plays an analogous role as  $\Gamma$ .

Coming back to the general case for  $\lambda_1$ , we get:

$$\boxed{\forall X \in \mathcal{X}_\pi, \quad J(X) = \langle g, H_{\frac{X}{\lambda_1}} \rangle = E\left(\frac{X}{\lambda_1}\right) + \frac{1}{2} \langle \Gamma, H''_{\frac{X}{\lambda_1}} \rangle}$$

## 5.2 $\Gamma$ is a non negative measure

To further characterize our class of performance measures, we consider simple financial instruments, assuming that one can enter in any digital contract, i.e. a contract whose payoff distribution is a Bernoulli variable. By assumption (R),  $\mathcal{X}_\pi$  contains at least the centered Bernoulli variables (then all Bernoulli variables, from proposition 4.7). To get a performance measure with financial consistency, we make the following assumption:

**Assumption (F):** *It cannot exist centered Bernoulli variables with an arbitrary large  $\pi$ .*

**Definition 5.1** *A performance measure  $\pi$  is called **admissible** if it satisfies assumptions (A), (R) and (F).*

**Theorem 5.2** *An admissible performance measure  $\pi$  can be written:*

$$\forall X \in \mathcal{X}_\pi, \quad \pi(X) = \sup_{\lambda} \mathcal{J}(\lambda X) \quad \text{with} \quad J(X) = E\left(\frac{X}{\lambda_1}\right) + \frac{1}{2} \langle \Gamma, H''_{\frac{X}{\lambda_1}} \rangle$$

where  $\Gamma$  is a non negative measure with compact support such that  $\langle \Gamma, 1 \rangle = 1$ .

$$\text{If } X \mapsto \lambda_X \text{ is an optimal weight function, we have:} \quad \begin{cases} E(X) = 0 \Rightarrow \pi(X) = 0 \\ E(X) > 0 \Rightarrow \lambda_X \geq 0 \\ E(X) < 0 \Rightarrow \lambda_X \leq 0 \end{cases}$$

A reciprocal property of this representation result is proved in Appendix C. Note that we have now:  $\forall X \in \mathcal{X}_\pi$ ,  $J(X) = \langle g, H_{\frac{X}{\lambda_1}} \rangle$  where  $g$  is a distribution of order at most 2, since the order of  $\Gamma$  is 0. In particular, any measure based on 3 or 4 first moments (see references in [24] for example) cannot be an admissible performance measure, since it involves the third or fourth order derivative of the log-Laplace.

Proof: we need to prove it only with  $\lambda_1 = 1$ .

1. From assumption (R),  $\mathcal{X}_\pi$  contains the centered Bernoulli variables, in particular, for  $p \in ]0, 1[$ , the variable

$$X = \begin{cases} 1 - p & \text{with probability } p \\ -p & \text{with probability } 1 - p \end{cases}$$

We prove in Appendix B that for given  $\lambda > 0, x_0 \neq 0$ , we have, for  $p = \frac{e^{\lambda x_0}}{1+e^{\lambda x_0}}$ :

$$H''_{\lambda X}(x) = -\lambda \varphi_{\lambda, x_0}(x) \text{ with } \varphi_{\lambda, x_0}(x) = \frac{\lambda e^{\lambda(x-x_0)}}{[1+e^{\lambda(x-x_0)}]^2} \text{ and } \int \varphi_{\lambda, x_0}(x) dx = 1.$$

When  $\lambda \rightarrow +\infty$ , the function  $\varphi_{\lambda, x_0}$  concentrates around  $x_0$ . With the same notations as in the previous section, we study the sign of  $\langle \Gamma, \varphi_{\lambda, x_0} \rangle$ . For  $L \in \mathbb{R}$ , we set  $a(L) = \inf_{x_0 \in \mathbb{R}, \lambda \geq L} \langle \Gamma, \varphi_{\lambda, x_0} \rangle$ . The function  $a$  is increasing. Let us prove that:

$$(5.5) \quad \lim_{L \rightarrow +\infty} a(L) \geq 0$$

By contradiction, if  $\lim_{L \rightarrow +\infty} a(L) \leq -b$ , with  $b > 0$ , then for arbitrary large  $L$ , there exists  $\lambda, x_0 \in \mathbb{R}$  such that  $\lambda \geq L$  and  $\langle \Gamma, \varphi_{\lambda, x_0} \rangle \leq -\frac{b}{2}$ . Since  $\pi(X) \geq J_1(\lambda X) = \lambda E(X) + \frac{1}{2} \langle \Gamma, H''_{\lambda X} \rangle$  and  $E(X) = 0$ , we get  $\pi(X) \geq -\frac{1}{2} \lambda \langle \Gamma, \varphi_{\lambda, x_0} \rangle \geq \frac{b}{4} \lambda \geq \frac{b}{4} L$ . This would lead to the existence of centered Bernoulli variables with arbitrary large  $\pi$ . Contradiction proves (5.5).

Setting  $\varphi_\lambda(x) = \varphi_{\lambda, 0}(x) = \frac{\lambda e^{\lambda x}}{(1+e^{\lambda x})^2}$ , we have  $\varphi_\lambda(x) = \varphi_\lambda(-x)$  and  $\varphi_{\lambda, x_0}(x) = \varphi_\lambda(x_0 - x)$ . Thus  $\forall x_0 \in \mathbb{R}$ ,  $\langle \Gamma, \varphi_{\lambda, x_0} \rangle = \Gamma * \varphi_\lambda(x_0)$  and  $a(L) = \inf_{x_0 \in \mathbb{R}, \lambda \geq L} \Gamma * \varphi_\lambda(x_0)$ . When  $\lambda \rightarrow +\infty$ ,  $\varphi_\lambda \rightarrow \delta_0$  in the distribution sense. Consequently the  $C^\infty$  function  $\Gamma * \varphi_\lambda$  converges toward  $\Gamma$  in the distribution sense. We set  $b(\lambda) = -\inf\{0, a(\lambda)\}$  for  $\lambda > 0$ . Then  $\lim_{\lambda \rightarrow +\infty} b(\lambda) = 0$  and the  $\Gamma * \varphi_\lambda + b(\lambda)$  are non negative  $C^\infty$  functions converging toward  $\Gamma$  in the distribution sense.

This proves that  $\Gamma$  is a non negative measure (a non negative distribution is a measure, see [27]).

2. Sign of an optimal weight function  $X \mapsto \lambda_X$ : from  $\langle \Gamma, H''_X \rangle$  non positive, we have for  $X \in \mathcal{X}_\pi$ ,  $\forall \lambda \in \mathbb{R}$ ,  $J(\lambda X) \leq \lambda E(X)$ . If  $E(X) = 0$ , we get  $J(\lambda X) \leq 0$  then  $\pi(X) = \max_\lambda J(\lambda X) = 0$ . If  $E(X) > 0$ , for any  $\lambda < 0$ , we have  $J(\lambda X) < -\frac{1}{2\lambda_1} \langle \Gamma, -H''_{\lambda X} \rangle \leq 0$ . Therefore  $\lambda_X \geq 0$ . If  $E(X) < 0$ , changing  $X$  in  $-X$ , we prove  $\lambda_X \leq 0$ .  $\square$

Our two basic examples are admissible measures (see Appendix C). We can verify that they correspond to distributions  $\Gamma$  which are non negative measures with compact support:

The half squared Sharpe ratio, with  $J_1(X) = E(X) + \frac{1}{2} H''_X(0)$ , corresponds to  $\Gamma_{Sh} = \delta_0$ ,  $g_{Sh} = -\delta'_0 + \frac{1}{2} \delta''_0$ , and  $\lambda_X = \lambda_1 \frac{E(X)}{V(X)}$  (for any  $\lambda_1 > 0$ ).

Hodges' measure, with  $J_1(X) = H_X(1)$  corresponds to  $g_{Hod} = \delta_1$ , up to a term  $c\delta_0$ . Following the construction of  $g, G, \Gamma$  in section 5.1, we take  $g = g_{Hod} = \delta_1 - \delta_0$  (from (5.3)). Then  $\frac{1}{2} \Gamma'' = g + \delta'_0 = \delta_1 - \delta_0 + \delta'_0$ . A primitive of  $\delta_a$  is  $\mathbb{1}_{x \geq a}$  and a primitive of  $\mathbb{1}_{x \geq a}$  is  $(x-a)\mathbb{1}_{x \geq a}$ . Then integrating twice with appropriate choice of constants, we get  $\frac{1}{2} \Gamma = (x-1)\mathbb{1}_{x \geq 1} - x\mathbb{1}_{x \geq 0} + 1_{x \geq 0}$  i.e. the Hodges measure corresponds to:

$$g_{Hod} = \delta_1 - \delta_0, \quad \Gamma_{Hod}(x) = 2(1-x)\mathbb{1}_{[0;1]}(x) \quad \text{and} \quad \lambda_X = \lambda_1 \argmax H_X.$$



## 6 Interpretation in terms of risk

We have for  $X \in \mathcal{X}_\pi$ ,  $J(X) = E(\frac{X}{\lambda_1}) + \frac{1}{2} \langle \Gamma, H''_{\frac{X}{\lambda_1}} \rangle$ . This formulation allows to separate the risk aversion preferences relative to two kinds of risk, gaussian and non-gaussian.

### 6.1 Gaussian risk aversion

**Lemma 6.1**  $\frac{1}{\lambda_1}$  corresponds to the usual risk aversion parameter for gaussian risks.

Proof: For  $X$  gaussian, we have  $J(X) = \frac{1}{\lambda_1} [E(X) - \frac{1}{2\lambda_1} V(X)]$ .

**Theorem 6.2** *If all investors face the same risks, then the relative proportions of the risky assets in any optimal portfolio are independent of gaussian risk preferences.*

Proof: from (4.7), in the optimal portfolio, the asset weights are proportional to those corresponding to  $\lambda_1 = 1$ , which is a known result in the mean-variance context.

We keep here a separability theorem for gaussian risks, as in the usual Capital Asset Pricing Model. The factor  $\lambda_1$  corresponds to the scale to which the investor considers the opportunities.

### 6.2 Non-gaussian risk aversion

For a given  $\lambda_1$  (say  $\lambda_1 = 1$ ), the remaining risk aversion is characterized by  $\Gamma$ . In  $J_1(X) = E(X) + \frac{1}{2} \langle \Gamma, H''_X \rangle$ , the second term appears as a term of risk:  $\rho(X) = \langle \Gamma, -H''_X \rangle$  is non negative and calculated on the centered random variable (since  $H''_{X-E(X)} = H''_X$ ). From  $\Gamma$  non negative and  $\langle \Gamma, 1 \rangle = 1$ ,  $\rho$  is a convex combination of the risk measures  $-H''(\lambda)$ , for  $\lambda$  in the support of  $\Gamma$ . Note that these risk measures are Esscher variances:  $-H''_X(\lambda)$  is the variance of  $X$  under the probability  $P_X^\lambda$  defined in (4.2). Therefore  $\rho$  corresponds to a **generalized variance**; it reduces to the classical variance if  $\Gamma = \delta_0$  (giving the half squared Sharpe ratio for  $\pi$ ).

Therefore our class of measures allows to develop a new CAPM, including non-gaussian risk aversion and efficient frontiers can be studied. This is more pertinent for monotones  $\pi$  (allowing convex frontiers) and this question will be addressed in further research, as will the properties of the risk measure  $\rho$ .

## 7 Practical application in an example of portfolio optimisation

We show how our family of measures will be better adapted than the Sharpe ratio in the presence of financial derivatives. We consider trading strategies on a future whose final net worth  $X$  is assumed to be gaussian and we calculate the optimal position when available assets are this future and all standard calls and puts on it. We recall that any final net payoff function  $s(X)$  can be achieved with these available trades.

We assume that the market is complete and that the risk-neutral and true probability distributions of  $X$  are gaussian with standard deviation 1. We denote by  $f^*$  the risk-neutral

density function (with mean 0) and by  $f$  the true density (with a mean  $m > 0$ ). We have for  $x \in \mathbb{R}$ ,  $f^*(x) = \frac{1}{2\pi}e^{-\frac{x^2}{2}}$  and  $f(x) = \frac{1}{2\pi}e^{-\frac{1}{2}(x-m)^2}$ . The Sharpe ratio is  $m$  for the future alone and we will take  $m = 2$  below.

We consider portfolios with final net worth  $s(X)$  satisfying  $E^*[s(X)] = 0$ , where  $E^*$  denotes the expectation under the risk-neutral probability, meaning that the portfolio is worth 0 at the beginning of the period. The optimal position is built successively according to three criteria: we optimize the Sharpe ratio, Hodges' measure and a third measure  $\pi_3$  obtained by choosing  $K$  with a cardinal equal to 2 (next simplest choice of admissible measure after Hodges' measure). Moreover the parameters of  $\pi_3$  are chosen in order to get a measure with good properties (see Bonnet and Nagot [14]), in particular  $\pi_3$  is monotone.

Note that if we denote by  $\mathcal{P}$  the set of available payoffs, since no a priori limit is set on leverage, solving  $\max_{X \in \mathcal{P}} \pi(X) = \max_{X \in \mathcal{P}} \max_{\lambda \in \mathbb{R}} J(\lambda X)$  is equivalent to solving  $\max_{X \in \mathcal{P}} J_1(X)$ . In conclusion, we solve, for each of the three criteria:

$$(7.1) \quad \max_{X \in \mathcal{P}} J_1(s(X)) \quad \text{under the constraint} \quad \int s(x)f^*(x)dx = 0$$

**1. According to the Sharpe ratio:**

We solve (7.1) with  $J_1(X) = E(X) - \frac{1}{2}V(X)$ . The optimal payoff  $s(X)$  maximizes

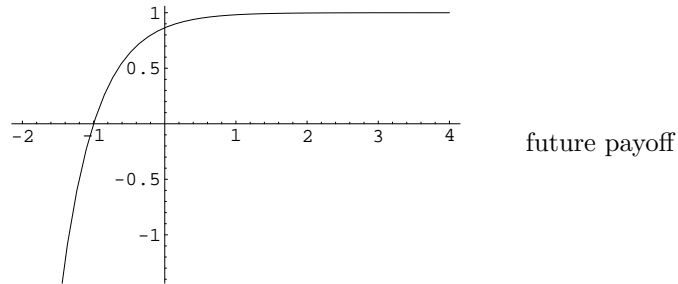
$$\int s(x)f(x)dx - \frac{1}{2} \left( \int s(x)^2 f(x)dx - \left[ \int s(x)f(x)dx \right]^2 \right)$$

The Euler-Lagrange conditions on  $s$  imply:

$$\forall x \in \mathbb{R}, \quad f(x) - \frac{1}{2} \left[ 2s(x)f(x) - 2f(x) \int s(y)f(y)dy \right] + cf^*(x) = 0$$

for some constant  $c$ . Therefore  $s(x) - c \frac{f^*(x)}{f(x)}$  is constant and  $s$  is proportional to  $1 - C \frac{f^*(x)}{f(x)}$ , with a constant  $C$  given by (7.1). We get  $C \int \frac{f^*(x)^2}{f(x)} dx = 1$  then  $C = e^{-m^2}$  and the optimal payoff is, up to a multiplicative constant:

$$s(x) = 1 - e^{-m^2} e^{-mx}, \quad \text{for } x \in \mathbb{R}$$

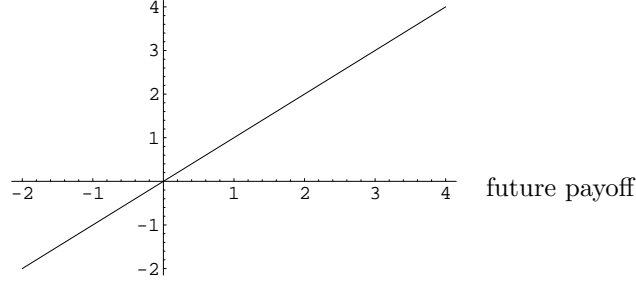


**2. According to Hodges' measure:**

We solve (7.1) with  $J_1(X) = H_X(1) = -\ln E[e^{-X}]$ . The optimal payoff  $s(X)$  maximizes

$$-\int e^{-s(x)} f(x)dx$$

Then there exists a constant  $C > 0$  such that  $\forall x \in \mathbb{R}, e^{-s(x)}f(x) - Cf^*(x) = 0$ . We get  $\forall x \in \mathbb{R}, s(x) = C_1 + \ln \frac{f(x)}{f^*(x)} = C_2 + mx$  and  $\int C_2 f^*(x)dx = 0$  implies  $C_2 = 0$ . We get  $s(x) = mx$  for  $x \in \mathbb{R}$ , i.e. the optimal portfolio corresponds to  $m$  units of the future.

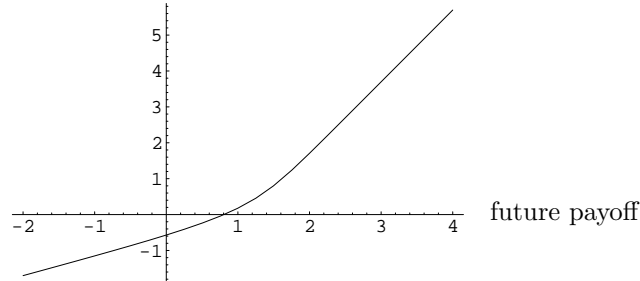


**3.** According to  $\pi_3$ :

We solve (7.1) with  $J_1(X) = \alpha_1 H_X(x_1) + \alpha_2 H_X(x_2)$ . We choose  $0 < x_1 < 1 < x_2$  to have a monotone measure and  $(\alpha_1, \alpha_2) = (\frac{x_2-1}{x_1(x_2-x_1)}, \frac{1-x_1}{x_2(x_2-x_1)})$  to get the right value on gaussian variables. The optimal payoff  $s(X)$  maximizes

$$-\sum_{i=1}^2 \alpha_i \ln \int e^{-x_i s(x)} f(x) dx$$

Then for some constants  $c_i$ , we have  $\forall x \in \mathbb{R}, \sum_{i=1}^2 c_i e^{-x_i s(x)} f(x) + c_3 f^*(x) = 0$  which gives  $\sum_{i=1}^2 C_i e^{-x_i s(x)} = e^{-mx}, \forall x \in \mathbb{R}$ , for some constants  $C_i$ . The function  $s$  can then be obtained as the inverse function of  $y \mapsto -\frac{1}{m} \ln[C_1 e^{-x_1 y} + C_2 e^{-x_2 y}]$ .



As described before, we find that the Sharpe ratio optimization leads to a quite dangerous choice, based on a short position in out-of-the-money puts, but scoring a Sharpe ratio of 7.32 (and 90 if the Sharpe ratio for the future is set to 3)! The Hodges' measure optimization proposes to be long in the future without any option position. The third choice has call-like characteristics.

## 8 Conclusion and future directions

A new class of performance measures is built, based on axioms designed for the alternative investment context and related to portfolio optimization. While coinciding with the squared

Sharpe ratio on gaussian variables, the new framework allows to analyse general final net worth distribution.

A full characterization of admissible measures is given, as the optimum of an associated measure  $J$  when varying the quantity of asset hold, this measure being a linear and continuous function of the logLaplace. Adding a financial assumption to rule out inappropriate measures, we express  $J$  as a Schwartz distribution of order at most two.

This expression helps to study the properties of the class of measures. In Bonnet and Nagot [14], further characterization in the case of monotone measures with respect to first or second order stochastic dominance are presented. We prove that both monotonicities are equivalent, and give the condition for them to hold. We then address the question of the unicity of the optimal weight function.

Preferences relationship associated to  $J$  is also studied in [14]. We prove that the axioms for a representation with a utility function are all satisfied except the independence axiom, which holds if and only if  $J$  corresponds to a CARA utility function. Associated risk measures are studied, addressing the question of coherence. These measures do not satisfy positive homogeneity but are convex.

Other further directions include the determination of the most appropriate measure for each business.

## Appendix A. Complement on Hodges' measure on $\mathcal{X}$

We discuss here the possibility of infinite values for  $Hod$  according to  $X$  being or not an arbitrage opportunity. Since we allow short selling, we take the following definition for an arbitrage opportunity:

**Definition 8.1** *An arbitrage opportunity is a variable  $X$  such that  $X \geq 0$   $P$ -ps, or  $X \leq 0$   $P$ -ps, and  $P(X = 0) < 1$ .*

**Notation 8.2** *We denote by  $S_X$  the closed convex envelop of the support of the law of  $X$ . Then  $S_X$  is a closed interval included in  $\overline{\mathbb{R}}$  and  $X \in \mathcal{X}^*$  is an arbitrage opportunity if and only if  $0 \notin \overset{\circ}{S}_X$  (i.e.  $S_X$  included in  $[0, +\infty]$  or  $[-\infty, 0]$ ).*

**Proposition 8.3** *We have  $Hod(0) = 0$  and for  $X \in \mathcal{X}^*$ :*

- (i) *If  $X$  is not an arbitrage opportunity,  $Hod(X) < +\infty$ ,  $\lambda_X$  is uniquely defined in  $\mathbb{R}$  and has the sign of  $E(X)$ .*
- (ii) *If  $X$  is a strict arbitrage opportunity ( $X > 0$   $P$ -ps or  $X < 0$   $P$ -ps), then  $Hod(X) = +\infty$ .*

This proves in particular that  $Hod(Z + X)$  is finite for any  $X \in \mathcal{X}$ , when  $Z$  is gaussian and non constant (from  $\overset{\circ}{S}_{Z+X} = \mathbb{R}$ ), which completes section 2.2.2.

Proof: we consider  $X \in \mathcal{X}^*$ .

(i) If  $X$  is not an arbitrage opportunity, we have  $0 \in \overset{\circ}{S}_X$ , then  $Hod(X) < +\infty$  (this is a classical consequence of Cramer's theorem). Since  $X$  is not constant,  $H_X$  is strictly concave and its maximum is achieved at a unique point  $\lambda_X \in D_X$  ( $H_X$  is worth  $-\infty$  outside  $D_X$ ). From  $H_X(0) = 0$ ,  $H'_X(0) = E(X)$  and  $H_X$  strictly concave, we deduce that  $\lambda_X$  and  $E(X)$  have same sign (and  $\lambda_X = 0$  if  $E(X) = 0$ ). Note that if  $H_X$  is steep<sup>13</sup>,  $\lambda_X \in \overset{\circ}{D}_X$ .

(ii) If for example  $X > 0$   $P$ -ps, then  $D_X$  contains  $[0, +\infty[$ . We have  $X \geq \varepsilon > 0$   $P$ -ps, and for  $\lambda > 0$ ,  $H_X(\lambda) \geq \varepsilon\lambda$  therefore  $Hod(X) = +\infty$  and  $\sup H_X$  is achieved at  $\lambda_X = +\infty$ .

## Appendix B.

We consider a Bernouilli variable  $X = \begin{cases} 1-p & \text{with probability } p \\ -p & \text{with probability } 1-p \end{cases}$

Then  $E(X) = 0$  and  $V(X) = p(1-p)^2 + (1-p)p^2 = p(1-p)$ . We have for  $x \in \mathbb{R}$ ,  $H_X(x) = -\ln [pe^{-(1-p)x} + (1-p)e^{px}]$  then

$$H'_X(x) = -\frac{-p(1-p)e^{-(1-p)x} + p(1-p)e^{px}}{pe^{-(1-p)x} + (1-p)e^{px}} = p(1-p)\frac{1-e^x}{p + (1-p)e^x},$$

$$H''_X(x) = p(1-p)\frac{-e^x [p + (1-p)e^x] - (1-p)e^x(1-e^x)}{[p + (1-p)e^x]^2} = -p(1-p)\frac{e^x}{[p + (1-p)e^x]^2}$$

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<sup>13</sup>I.e., for any boundary point of  $D_X$ ,  $\lambda_0$ ,  $\lim_{\lambda \rightarrow \lambda_0, \lambda \in \overset{\circ}{D}_X} |H'_X(\lambda)| = +\infty$ .

and

$$H_X''(x) = -\frac{p}{1-p} \frac{e^x}{\left[\frac{p}{1-p} + e^x\right]^2} \leq 0.$$

For given  $\lambda > 0$ ,  $x_0 \in \mathbb{R}$ , we choose  $p$  such that  $\ln \frac{p}{1-p} = \lambda x_0$  i.e.  $p = \frac{e^{\lambda x_0}}{1+e^{\lambda x_0}} \in ]0, 1[$ .

Then  $H_{\lambda X}''(x) = \lambda^2 H_X''(\lambda x) = -\lambda^2 e^{\lambda x_0} \frac{e^{\lambda x}}{[e^{\lambda x_0} + e^{\lambda x}]^2} = -\lambda^2 \frac{e^{\lambda(x-x_0)}}{[1+e^{\lambda(x-x_0)}]^2} = -\lambda \varphi_{\lambda, x_0}(x)$  where

$$\varphi_{\lambda, x_0}(x) = \frac{\lambda e^{\lambda(x-x_0)}}{[1+e^{\lambda(x-x_0)}]^2}. \text{ Note that } \int \varphi_{\lambda, x_0}(x) dx = \int \frac{e^y}{[1+e^y]^2} dy = \left[ \frac{-1}{1+e^y} \right]_{-\infty}^{+\infty} = 1.$$

When  $\lambda \rightarrow +\infty$ ,  $p$  converges toward 1 if  $x_0 > 0$ , 0 if  $x_0 < 0$ , while  $p = \frac{1}{2}$  if  $x_0 = 0$ . In any case the function  $\varphi_{\lambda, x_0}$  concentrates around  $x_0$  when  $\lambda \rightarrow +\infty$ . Precisely,  $\varphi_{\lambda, x_0}$  converges toward  $\delta_{x_0}$  in the distribution sense.

### Appendix C. Sufficient condition

We establish a reciprocal property of theorem 5.2. We consider a non negative measure  $\Gamma$  with compact support  $K$  such that  $\langle \Gamma, 1 \rangle = 1$  and  $\pi$  given by:

$$\forall X \in \mathcal{X}, \pi(X) = \sup_{\lambda \in \mathbb{R}} J(\lambda X) \text{ with } J(X) = \begin{cases} E(X) + \frac{1}{2} \langle \Gamma, H_X'' \rangle & \text{if } X \in \mathcal{X}_K \\ -\infty & \text{if } X \notin \mathcal{X}_K \end{cases}$$

Replacing  $K$  by the the convex envelop of  $K \cup \{0\}$ , we get a closed interval containing 0 (without changing the definition of  $J$  since  $\mathcal{X}_K$  does not change). We set  $\mathcal{J}(H) = \langle -\delta_0' + \frac{1}{2} \Gamma'', H \rangle$  for  $H \in C^\infty(K)$ . Then  $\mathcal{J}$  is linear and continuous for the semi-norm  $\|\cdot\|_{K,2}$  and  $\forall X \in \mathcal{X}_K$ ,  $\mathcal{J}(H_X) = J(X)$ .

Notes: • A scaling factor  $\lambda_1$  could be incorporated, to reflect different gaussian risk aversions, by setting  $\pi(X) = \sup_{\lambda \in \mathbb{R}} J(\frac{\lambda}{\lambda_1} X)$  for  $X \in \mathcal{X}$ .

• For  $X \in \mathcal{X}$ , we have  $\pi(X) = \sup_{\lambda/\lambda X \in \mathcal{X}_K} \mathcal{J}(H_{\lambda X})$ , upper bound on an open set which contains a neighborhood of 0, from Proposition 4.2. Moreover  $\pi(X)$  is non negative.

• This will not give exactly the reciprocal property, since in section 5, we can have  $X \in \mathcal{X}_K$  and  $J(X) = -\infty$ , which is not the case here, in particular  $\mathcal{X}_K$  and  $\mathcal{X}_\pi$  will coincide.

**Proposition 8.4**  $\pi$  is admissible.

This proves in particular that both examples of section 2.2 satisfy property (R).

Proof: • Law invariance is obvious since  $J(X)$ , then  $\pi(X)$ , depends only on  $H_X$ .

•  $\pi$  satisfies assumption (A1).

$J$  is additive on  $\mathcal{X}_K$ . Thus for  $X, Y \in \mathcal{X}$  independent, (2.1) comes from

$$(8.1) \quad \sup_{\alpha, \beta \in \mathbb{R}} \sup_{\lambda \in \mathbb{R}} J(\lambda(\alpha X + \beta Y)) = \sup_{\alpha, \beta \in \mathbb{R}} J(\alpha X + \beta Y) = \sup_{\alpha \in \mathbb{R}} J(\alpha X) + \sup_{\beta \in \mathbb{R}} J(\beta Y)$$

- $\pi$  satisfies assumption **(A2)**.

From the continuity of  $\mathcal{J}$  and Proposition 4.2,  $\lambda \mapsto \mathcal{J}(H_{\lambda X})$  is continuous on  $\{\lambda \in \mathbb{R} | \lambda X \in \mathcal{X}_K\}$ . But due to the existence of variables  $X$  with particular behavior of  $H_X$  at the boundary of  $D_X$ , the upper bound defining  $\pi(X)$  is not necessary achieved. For example in the Hodges' measure case,  $\pi(X) = \sup H_X$  and this upper bound can be achieved on the boundary of  $D_X$ : we can have  $D_X = [-1, 1]$ ,  $\sup H_X = \lim_{\substack{\lambda \rightarrow 1 \\ \lambda < 1}} H_X(\lambda)$  while  $J(X) = H_X(1) = -\infty$  (non steep  $H_X$ , for an example see [8]). To take account of these limit cases, we replace  $J$  by  $\bar{J}$  defined by:

$$(8.2) \quad \bar{J}(X) = \overline{\lim_{\substack{\lambda \rightarrow 1 \\ \lambda < 1}} J(\lambda X)}, \text{ for } X \in \mathcal{X}$$

$\lambda \mapsto -\bar{J}(\lambda X)$  corresponds to the lower semicontinuous closure of  $\lambda \mapsto -J(\lambda X)$  (see [25]). From the continuity of  $\lambda \mapsto J(\lambda X)$ ,  $J$  and  $\bar{J}$  coincide on  $\mathcal{X}_K$ . If  $\lambda X \notin \mathcal{X}_K$  for  $\lambda$  in a neighborhood of 1,  $J(X) = \bar{J}(X) = -\infty$ . Therefore  $J(X)$  and  $\bar{J}(X)$  can differ only if  $K$  and  $D_X$  have a common extremity. In that case  $J(X) = -\infty$  while  $\bar{J}(X)$  can be finite.

For  $X \in \mathcal{X}$ , we have

$$\pi(X) = \sup_{\lambda \in \mathbb{R}} J(\lambda X) = \sup_{\lambda \in \mathbb{R}} \bar{J}(\lambda X)$$

and the last upper bound is achieved on  $\overline{\mathbb{R}}$ : we get  $\lambda_X \in \mathbb{R}$  such that  $\pi(X) = \bar{J}(\lambda_X X)$ , or  $\lambda_X = +\infty$  (resp.  $-\infty$ ) if  $\pi(X) = \lim_{\lambda \rightarrow +\infty} J(\lambda X)$  (resp.  $\lim_{\lambda \rightarrow -\infty}$ ). Together with (8.1) (written with  $J$  replaced by  $\bar{J}$ , still additive), this leads to (A2), with these  $\{\lambda_X\}$ .

- $\pi$  satisfies assumption **(AT)**.

For  $X$  gaussian, we have  $X \in \mathcal{X}_K$  and  $J(X) = E(X) - \frac{1}{2}V(X)$ , then  $\lambda_X = \frac{E(X)}{V(X)}$  and  $\pi(X) = \frac{1}{2}Sh^2(X)$ , which proves  $\pi \not\equiv 0$  on gaussian variables,  $\lambda_Z = 1$  for  $Z \sim \mathcal{N}(1, 1)$  and  $x \mapsto \pi(\mathcal{N}(x, x))$  bounded on a non empty open interval.

Moreover for  $X \in \mathcal{X}$  and any non constant gaussian variable  $Z$ , we have  $\pi(Z + X) < +\infty$ . Indeed  $\forall \lambda \in \mathbb{R}$ ,  $J(\lambda(Z + X)) \leq \lambda[E(X) + E(Z)] - \frac{1}{2}\lambda^2 V(Z)$  (from  $\langle \Gamma, H''_{\lambda(Z+X)} \rangle \leq \langle \Gamma, H''_{\lambda Z} \rangle$ , since  $\langle \Gamma, H''_{\lambda X} \rangle \leq 0$ ). Then for  $V(Z) > 0$ ,  $\sup_{\lambda} J(\lambda(Z + X))$  is achieved on  $\mathbb{R}$  and is finite.

- $\pi$  satisfies assumption **(F)**. Indeed, for any centered variable  $X$ , if  $\lambda X \in \mathcal{X}_K$  (e.g.  $X$  Bernouilli), we have, since  $\Gamma$  is non negative,  $J(\lambda X) = \frac{1}{2}\langle \Gamma, H''_{\lambda X} \rangle \leq 0$ , then  $\pi(X) = 0 < +\infty$ .

- $\pi$  satisfies assumption **(R)**.

For  $x > 0$ ,  $Z_x$  defined in notation 2.4 (with  $\lambda_1 = 1$ ) satisfies  $\pi(Z_x) = x$ ,  $\lambda_{Z_x} = 1$  and for  $\lambda \in \mathbb{R}$ ,  $J(\lambda Z_x) = -x(\lambda^2 - 2\lambda)$ , since  $\lambda Z_x \sim \mathcal{N}(2x\lambda, 2x\lambda^2)$ . For  $X \in \mathcal{X}^{14}$  and  $x > 0$ , we have then  $\pi(Z_x + X) - \pi(Z_x) = \sup_{\lambda \in \mathbb{R}} [J(\lambda X) + J(\lambda Z_x)] - x = \sup_{\lambda \in \mathbb{R}} f_x^X$  (upper bound on  $\mathbb{R}$ ), with  $f_x^X(\lambda) = J(\lambda X) - x(\lambda - 1)^2$ . This proves that  $x \mapsto \pi(Z_x + X) - \pi(Z_x)$  is non increasing on  $\mathbb{R}^+$ .

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<sup>14</sup>independent from all  $Z_x, x > 0$ .

**Lemma 8.5** For  $X \in \mathcal{X}$ , if  $\lim_{x \rightarrow +\infty} \searrow [\pi(Z_x + X) - \pi(Z_x)] \in \mathbb{R}$ , then  $\lim_{x \rightarrow +\infty} \lambda_{Z_x + X} = 1$ .

Proof: Let  $X \in \mathcal{X}$  such that  $L = \lim_{x \rightarrow +\infty} [\pi(Z_x + X) - \pi(Z_x)] \in \mathbb{R}$ . If we prove  $\lambda_{Z_x + X} \xrightarrow{x \rightarrow +\infty} 1$  when  $\pi(X) < +\infty$ , then, for other cases, since  $\pi(Z_1 + X) < \infty$ , we will have  $\lambda_{Z_{x+1} + X} = \lambda_{Z_x + Z_1 + X} \xrightarrow{x \rightarrow +\infty} 1$  and the conclusion will still hold. We assume then  $\pi(X) < +\infty$ .

For  $\eta > 0$  and  $|\lambda - 1| \geq \eta$ , we have  $f_x^X(\lambda) \leq J(\lambda X) - x\eta^2 \leq \pi(X) - x\eta^2$ . Considering  $x$  large enough, we get  $\pi(X) - x\eta^2 \leq L - 1$  and  $\sup_x f_x^X > L - 1$ , since  $L = \inf_x \sup_x f_x^X$ . Thus  $\sup_x f_x^X$  is achieved on  $[1 - \eta, 1 + \eta]$ , which then contains  $\lambda_{Z_x + X}$ .

**Lemma 8.6** The set  $\mathcal{X}_\pi$  defined as in notation 2.5, with  $\lambda_1 = 1$ , satisfies  $\mathcal{X}_\pi \subset \mathcal{X}_K$ .

Proof: For  $X \in \mathcal{X}_\pi$ ,  $\lim_{x \rightarrow +\infty} [\pi(Z_x + X) - \pi(Z_x)] \in \mathbb{R}$ , then from previous lemma, for  $x$  large enough, there exists  $\lambda$  arbitrarily close to 1 such that  $J(\lambda(Z_x + X)) \in \mathbb{R}$  (since  $\pi(Z_x + X) \in \mathbb{R}$ ), implying  $\lambda(Z_x + X) \in \mathcal{X}_K$ . Since  $D_{\lambda(Z_x + X)} = D_{\lambda X}$  for  $\lambda > 0$ , we get  $\lambda X \in \mathcal{X}_K$  with  $\lambda$  arbitrarily close to 1. We could yet have  $X \notin \mathcal{X}_K$ , but the same argument for  $\alpha X$  replacing  $X$ , with  $\alpha > 1$  arbitrarily close to 1 (from the definition of  $\mathcal{X}_\pi$  and lemma 8.5, we still have  $\lim_{x \rightarrow +\infty} \lambda_{Z_x + \alpha X} = 1$ ) allows to prove  $X \in \mathcal{X}_K$ . Then  $\mathcal{X}_\pi \subset \mathcal{X}_K$  (or  $K \subset \bigcap_{X \in \mathcal{X}_\pi} \overset{\circ}{D}_X$ ).

**Lemma 8.7** For  $X_1, \dots, X_k \in \mathcal{X}_K$ ,  $\forall \eta > 0$ ,  $\exists M_\eta > 0$  such that  $\forall H_Y \in \text{Vect}_{\mathbb{N}}(H_{X_1}, \dots, H_{X_k})$ ,  $x \geq M_\eta \|H_Y\|_{K,2}$  implies  $\lambda_{Z_x + Y} \in [1 - \eta, 1 + \eta]$ .

Proof: Let  $X_1, \dots, X_k \in \mathcal{X}_K$ . We set  $m = \min_{1 \leq i \leq k} J(X_i) \in \mathbb{R}$  and  $M = \max_{1 \leq i \leq k} \pi(X_i) \in [0, +\infty]$ .

1. We assume  $M < +\infty$ . Let  $n_1, \dots, n_k \in \mathbb{N}$  and  $H_Y = n_1 H_{X_1} + \dots + n_k H_{X_k}$ . We have  $\sup_x f_x^Y \geq f_x^Y(1) = \sum_i n_i J(X_i) \geq n_Y m$  where  $n_Y = \sum_{1 \leq i \leq k} n_i$ .

Let  $\eta > 0$ . For  $|\lambda - 1| \geq \eta$ , we have  $f_x^Y(\lambda) \leq \sum_i n_i J(\lambda X_i) - x\eta^2 \leq n_Y M - x\eta^2$ . Assuming  $x \geq [M - (m - 1)] \frac{n_Y}{\eta^2}$  (note that  $m \leq M$ ), we get  $f_x^Y(\lambda) \leq n_Y(m - 1)$  for  $\lambda \notin [1 - \eta, 1 + \eta]$ . Thus  $\pi(Z_x + Y) - \pi(Z_x) = \sup_x f_x^Y = \sup_{\lambda \in [1 - \eta, 1 + \eta]} f_x^Y(\lambda)$  and this upper bound is achieved at  $\lambda_{Z_x + Y}$ .

We have proved that  $\forall \eta > 0$ ,  $\exists M_\eta > 0$  such that  $\forall n_1 H_{X_1} + \dots + n_k H_{X_k} \in \text{Vect}_{\mathbb{N}}(H_{X_1}, \dots, H_{X_k})$ , for  $x \geq M_\eta \sum_{1 \leq i \leq k} n_i$ , we have  $\lambda_{Z_x + Y} \in [1 - \eta, 1 + \eta]$ .

2. General case. We set  $\mathcal{I} = \{1 \leq i \leq k \mid X_i \text{ is not constant}\}$ . Any  $H_Y \in \text{Vect}_{\mathbb{N}}(H_{X_1}, \dots, H_{X_k})$  can be written  $H_Y = n_0 H_{X_0} + \sum_{i \in \mathcal{I}} n_i H_{X_i}$  with  $n_0 \in \mathbb{R}$  and  $X_0$  constant equal to 1. Let  $n_Y = n_0 + \sum_{i \in \mathcal{I}} n_i$  (not uniquely defined). For  $x > 0$  and  $\forall \lambda \in \mathbb{R}$ , we have  $H_{Z_{n_Y} + Y} = \sum_{i \in \mathcal{I} \cup \{0\}} n_i H_{Z_1 + X_i}$  and  $\pi(Z_1 + X_i) < +\infty$  for  $i \in \mathcal{I} \cup \{0\}$ . Then  $\forall \eta > 0$ ,  $\exists M_\eta > 0$  such that  $\forall H_Y \in \text{Vect}_{\mathbb{N}}(H_{X_1}, \dots, H_{X_k})$ , for  $x \geq M_\eta n_Y$  (for at least one of the possible values for  $n_Y$ ) we have  $\lambda_{Z_x + n_Y + Y} \in [1 - \eta, 1 + \eta]$ .

Now we link  $n_Y$  to  $\|H_Y\|_{K,2}$ . We have  $\|H_Y\|_{K,2} \geq \max_K |H_Y''| \geq \min_K |H_Y''| = \min_K \sum_{i \in \mathcal{I}} n_i (-H_{X_i}'')$



since for  $i \in \mathcal{I}$ ,  $H''_{X_i}$  is negative ( $H_X$  is strictly concave for  $X$  non constant). Thus, with  $A = \min_{i \in \mathcal{I}, \lambda \in K} [-H''_{X_i}(\lambda)] > 0$ , we have:

$$(8.3) \quad A \sum_{i \in \mathcal{I}} n_i \leq \|H_Y\|_{K,2}$$

But  $|n_0| = |H'_Y - \sum_{i \in \mathcal{I}} n_i H'_{X_i}| \leq \max_K |H'_Y| + A' \sum_{i \in \mathcal{I}} n_i$  with  $A' = \max_{i \in \mathcal{I}} \max_K |H'_{X_i}|$ . Using (8.3), we get  $|n_0| \leq (1 + \frac{A'}{A}) \|H_Y\|_{K,2}$  and by (8.3) again,  $n_Y \leq C \|H_Y\|_{K,2}$ , with  $C$  independent of  $Y$ .

In conclusion, for  $x \geq C(M_\eta + 1) \|H_Y\|_{K,2}$ , since  $x - n_Y \geq M_\eta n_Y$ , we have  $\lambda_{Z_n+Y} \in [1 - \eta, 1 + \eta]$  and the lemma is proved.  $\square$

To complete the proof of assumption (R), let us consider  $X_1, \dots, X_k \in \mathcal{X}_K$ . The open set  $I = \cap_{1 \leq i \leq k} \overset{\circ}{D}_{X_i}$  contains  $K$ , let  $\eta_0 \in ]0, 1[$  such that  $K^{\eta_0} \subset I$ . We consider  $0 < \eta \leq \eta_0$ , then  $K \subset K^\eta \subset I$ .

For  $H_Y, H_{Y'} \in Vect_{\mathbb{N}}(H_{X_1}, \dots, H_{X_k}) \subset \mathcal{H}_I$  and  $\lambda \in [1 - \eta, 1 + \eta]$ , we have  $\lambda Y, \lambda Y' \in \mathcal{X}_K$ , since  $\mathcal{X}_I \subset \mathcal{X}_{K^\eta} \subset \mathcal{X}_{\lambda K}$ . From the continuity of  $\mathcal{J}$  on  $C^\infty(K)$ , we get:

$$(8.4) \quad |J(\lambda(Z_n + Y)) - J(\lambda(Z_n + Y'))| = |\mathcal{J}(H_{\lambda Y}) - \mathcal{J}(H_{\lambda Y'})| \leq \|\mathcal{J}\| \|H_{\lambda Y} - H_{\lambda Y'}\|_{K,2}$$

Since  $H_{\lambda X}^{(k)}(\cdot) = \lambda^k H_X^{(k)}(\lambda \cdot)$ , if  $\lambda \in [1 - \eta, 1 + \eta]$ , we have:

$$(8.5) \quad \|H_{\lambda Y} - H_{\lambda Y'}\|_{K,2} = \max_{0 \leq k \leq 2} [\lambda^k \max_{\lambda K} |H_Y^{(k)} - H_{Y'}^{(k)}|] \leq (1 + \eta)^2 \|H_Y - H_{Y'}\|_{K^\eta,2}$$

From lemma 8.7, there exists  $M_\eta$  such that if  $n \geq M_\eta \|H_Y\|_{K,2}$ , then  $\lambda_{Z_n+Y} \in [1 - \eta, 1 + \eta]$ . In that case  $\lambda_{Z_n+Y} K \subset I \subset \overset{\circ}{D}_Y = \overset{\circ}{D}_{Z_n+Y}$  and since  $J$  and  $\bar{J}$  coincide at  $\lambda_{Z_n+Y}(Z_n + Y) \in \mathcal{X}_K$ , we get  $\pi(Z_n + Y) = J(\lambda_{Z_n+Y}(Z_n + Y)) \in \mathbb{R}$ . Using (8.4) and (8.5) with  $\lambda = \lambda_{Z_n+Y}$ , we get:

$$|\pi(Z_n + Y) - J(\lambda_{Z_n+Y}(Z_n + Y'))| \leq M \|H_Y - H_{Y'}\|_{K^\eta,2}, \text{ with } M = (1 + \eta)^2 \|\mathcal{J}\|.$$

Then  $\pi(Z_n + Y') \geq J(\lambda_{Z_n+Y}(Z_n + Y')) \geq \pi(Z_n + Y) - M \|H_Y - H_{Y'}\|_{K^\eta,2}$  and by symmetry, we get for  $n \geq M_\eta \max(\|H_Y\|_{K,2}, \|H_{Y'}\|_{K,2})$ :

$$(8.6) \quad |\pi(Z_n + Y) - \pi(Z_n + Y')| \leq M \|H_Y - H_{Y'}\|_{K^\eta,2}$$

Then for  $X_1, \dots, X_k \in \mathcal{X}_K$  (which contains  $\mathcal{X}_\pi$ ), we have found  $\eta_0$  such that  $\forall \eta \leq \eta_0$ ,  $\exists M_\eta$  such that  $\forall H_Y, H_{Y'} \in Vect_{\mathbb{N}}(H_{X_1}, \dots, H_{X_k})$ ,

$$n \geq M_\eta \max(\|H_Y\|_{K,2}, \|H_{Y'}\|_{K,2}) \Rightarrow |\pi(Z_n + Y) - \pi(Z_n + Y')| \leq M \|H_Y - H_{Y'}\|_{K^\eta,2}.$$

We get that (R) is satisfied with  $p = 2$  and this completes the proof of proposition 8.4.

Note that we have in fact  $\mathcal{X}_\pi = \mathcal{X}_K$ . Indeed, for  $X \in \mathcal{X}_K$  and  $\alpha$  in a neighborhood

of 1, we have  $\alpha X \in \mathcal{X}_K$  and from (8.6) with  $Y = X_1 = \alpha X$ ,  $Y' = 0$ , for  $\eta$  small and  $n$  large enough:  $|\pi(Z_n + \alpha X) - \pi(Z_n)| \leq M \|H_{\alpha X}\|_{K^\eta, 2}$  while  $K^\eta \subset \overset{\circ}{D}_{\alpha X}$ . Therefore  $\lim_{n \rightarrow +\infty} [\pi(Z_n + \alpha X) - \pi(Z_n)] \in \mathbb{R}$ , then  $X \in \mathcal{X}_\pi$  and  $\mathcal{X}_\pi = \mathcal{X}_K$  from lemma 8.6.

Exemple: the assumption (R) is satisfied with  $K = \{0\}$  and  $p = 2$  for the squared Sharpe ratio (note that  $\mathcal{X} = \mathcal{X}_K$  and  $K^\eta = \{0\}$ ), and with  $K = [1 - \varepsilon, 1 + \varepsilon]$  and  $p = 2$  for Hodges measure, with  $\varepsilon$  arbitrary small.

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